

HARNACK'S INEQUALITY AND A PRIORI ESTIMATES FOR FRACTIONAL POWERS OF NON-SYMMETRIC DIFFERENTIAL OPERATORS

H. AIMAR, G. BELTRITTI, I. GÓMEZ, AND C. RIOS

ABSTRACT. We obtain a new general extension theorem in Banach spaces for operators which are not required to be symmetric, and apply it to obtain Harnack estimates and a priori regularity for solutions of fractional powers of several second order differential operators. These include weighted elliptic and subelliptic operators in divergence form (nonnecessarily self-adjoint), and nondivergence form operators with rough coefficients. We utilize the reflection extension technique introduced by Caffarelli and Silvestre.

Date: October 12, 2016.

2000 Mathematics Subject Classification. 35J70, 35B45, 35B60, 35B65, 35C15, 35H20.

Key words and phrases. Dirichlet to Neumann, functional calculus, Harnack's inequality, elliptic operators, Dirichlet forms, nondivergence.

1. INTRODUCTION

We consider several nonnegative second order differential operators L densely defined on a Banach space. Under different structural assumptions we will establish Harnack estimates for nonnegative solutions of fractional powers of L as consequence of existing Harnack estimates for an associated extended problem. This technique was pioneered by Caffarelli and Silvestre [3] for the fractional Laplacian, and it has already been multiplied into a great number of applications. In a nutshell, this is how the technique works to obtain results for the square root of for the Laplacian operator: if $u(x, y)$ is the smooth bounded solution of the extension problem

$$\begin{aligned} u(x, 0) &= f(x) && \text{for } x \in \mathbb{R}^n, \\ \Delta u(x, y) &= 0 && \text{for } x \in \mathbb{R}^n \text{ and } y > 0, \end{aligned}$$

then that $(-\Delta_x)^{1/2} f(x) = -u_y(x, 0)$ (the Dirichlet to Neumann map) as it has long been known. On the other hand, if $(-\Delta_x)^{1/2} f \equiv 0$ in an open set $\Omega \subset \mathbb{R}^n$, then the extended function

$$u(x, y) = \begin{cases} u(x, y) & y \geq 0 \\ u(x, -y) & y < 0 \end{cases}$$

is a solution of $\Delta u = 0$ in $\Omega \times \mathbb{R}$. If moreover $f \geq 0$ in Ω it follows (by Poisson's formula) that $u \geq 0$ in $\Omega \times \mathbb{R}$ and therefore u satisfies Harnack's inequality there. As a consequence, $f(x) = u(x, 0)$ satisfies Harnack's inequality in Ω , thus obtaining Harnack's estimates for nonnegative solutions of $(-\Delta_x)^{1/2} f = 0$. This principle was extended in [3] to other powers σ of the Laplacian by considering the extension problem

$$\begin{aligned} u(x, 0) &= f(x) && \text{on } \mathbb{R}^n \\ -\Delta_x u + \frac{1-2\sigma}{y} u_y + u_{yy} &= 0 && \text{in } \mathbb{R}^n \times (0, \infty), \end{aligned}$$

and proving that for constants $c_{\sigma,1}, c_{\sigma,2}$

$$\lim_{y \rightarrow 0^+} \frac{u(x, y) - u(x, 0)}{y^{2\sigma}} = c_{\sigma,1} (-\Delta)^{\sigma} f(x) = c_{\sigma,2} \lim_{y \rightarrow 0^+} y^{1-2\sigma} u_y(x, y).$$

The operator $-\Delta_x + \frac{1-2\sigma}{y} \frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2} = -\operatorname{div}_{(x,y)} (y^{1-2\sigma} \nabla_{(x,y)} \cdot)$ satisfies a Harnack inequality as consequence of the Fabes-Kenig-Serapioni results for weighted elliptic operators [10], and the result follows for powers $0 < \sigma < 1$ in the same way as for $\sigma = \frac{1}{2}$.

For a general operator L defined in \mathbb{R}^n this procedure may be divided in three main steps:

- (1) Solving the extension problem in \mathbb{R}_+^{n+1} . Find a solution of

$$\begin{aligned} \mathcal{L}u &= \left(-L_x + \frac{1-2\sigma}{y}\partial_y + \partial_y^2\right)u = 0 & (x, y) \in \mathbb{R}^n \times (0, \infty), \\ u(x, 0) &= f(x) & x \in \mathbb{R}^n. \end{aligned}$$

Prove that

$$\lim_{y \rightarrow 0^+} \frac{u(x, y) - u(x, 0)}{y^{2\sigma}} = c_{\sigma,1} L^\sigma f(x) = c_{\sigma,2} \lim_{y \rightarrow 0^+} y^{1-2\sigma} u_y(x, y).$$

- (2) If $Lf \equiv 0$ in an open set $\Omega \subset \mathbb{R}^n$, show that the extended function

$$u(x, y) = \begin{cases} u(x, y) & y \geq 0 \\ u(x, -y) & y < 0 \end{cases}$$

is a solution of $\mathcal{L}u = 0$ in $\Omega \times \mathbb{R}$.

- (3) Establish (from existing literature or otherwise) that nonnegative solutions to $\mathcal{L}u = 0$ satisfy a Harnack's inequality, or have a priori estimates which, by restriction, are also valid for solutions to $L^\sigma f = 0$.

In [27] Torrea and Stinga established (1) for a very general class of second order self-adjoint linear differential operators, and applied the technique to obtain Harnack's estimates for solutions of the fractional harmonic oscillator operator $H^\sigma = \left(-\Delta + |x|^2\right)^\sigma$.

In this work we extend (1) to operators that might not be self-adjoint, and apply the results to a variety of important examples. In particular, the application of the extension techniques and the existence of the functional calculus to nondivergence form operators is new in this level of generality. The paper is organized as follows. In the remainder of this introduction we list three different applications of our main extension theorem. In Section 2 we state and prove our main result, the extension theorem for closed operators on Banach spaces. In Section 3 we prove the three applications presented here. We note that our result for subelliptic operators in Section 3.2 includes a bigger class of operators than the diagonal ones presented in Theorem 1.4 below, and that Theorems 1.1 and 1.5 may also be extended to operators with drift and zero order terms. Finally, in the Appendix, Section 4, we include some basic facts about non-symmetric Dirichlet forms and functional calculus for easier reference.

Our first application to illustrate the utility of our main extension theorem is to weighted elliptic operators. Given $0 < \lambda \leq \Lambda < \infty$, let $\mathcal{F}_n(\lambda, \Lambda)$ denote the set of all $n \times n$ real valued matrix functions $\mathbf{A}(x)$ such that

$$(1.1) \quad \mathbf{A}(x) \xi \cdot \xi \geq \lambda |\xi|^2 \quad \text{and} \quad |\mathbf{A}(x) \xi \cdot \eta| \leq \Lambda |\xi| |\eta| \quad \text{for all } x, \xi, \eta \in \mathbb{R}^n,$$

that is, $\mathcal{F}_n(\lambda, \Lambda)$ is the set of real valued $n \times n$ matrices which eigenvalues lie in the interval $[\lambda, \Lambda]$.

A weight w in the Muckenhoupt class A_2 is a nonnegative locally integrable function in \mathbb{R}^n such that

$$[w]_{A_2} = \sup_{x \in \mathbb{R}^n, r > 0} \left(\frac{1}{|B_r(x)|} \int_{B_r(x)} w(y) dy \right)^{\frac{1}{2}} \left(\frac{1}{|B_r(x)|} \int_{B_r(x)} \frac{1}{w(y)} dy \right)^{\frac{1}{2}} < \infty,$$

where $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$.

Given $\mathbf{A} \in \mathcal{F}_n(\lambda, \Lambda)$ and $w \in A_2$ there is an associated weighted elliptic operator $L_{\mathbf{A}, w} = L_w : \mathcal{D}(L_w) \rightarrow L^2(w)$ given by

$$L_w u(x) = -\frac{1}{w(x)} \operatorname{div} w(x) \mathbf{A}(x) \nabla u(x)$$

in the weak sense. Such operators are closed and sectorial, and so they have a holomorphic functional calculus which, in particular, enables the definition of fractional powers $(L_w)^\sigma$.

Theorem 1.1. *Let $0 < \sigma \leq 1$, for every $\mathbf{A} \in \mathcal{F}_n(\lambda, \Lambda)$ and $w \in A_2$ there exists a constant $M = M(\lambda, \Lambda, [w]_{A_2}, \sigma)$ such that if Ω is an open set in \mathbb{R}^n , $u \in \mathcal{D}(L^\sigma) \subset L^2(\Omega, w)$ and $(L_w)^\sigma u = 0$ in an open set $\Omega' \subseteq \Omega$, we have that whenever $B_{2r}(x) \subset \Omega'$*

$$\max_{B_r(x_0)} |u(x)| \leq \frac{M}{w(B_{2r}(x_0))} \|u\|_{L^2(w, \Omega)}^2.$$

Moreover, if u is nonnegative, then whenever $B_{2r}(x) \subset \Omega'$

$$\sup_{B_r(x)} u \leq M \inf_{B_r(x)} u.$$

In particular, solutions of $(L_w)^\sigma u = 0$ in Ω' are locally Hölder continuous in Ω' .

Another consequence of the extension technique applied to weighted elliptic operators and the Fabes-Kenig-Serapioni boundary Harnack we also obtain, in the same way as in [3] (Theorem 5.3), boundary Hölder continuity for solutions to fractional powers of L_w . We present this result to showcase the applicability of our extension theorem.

Theorem 1.2. *Let $0 < \sigma \leq 1$, for every $\mathbf{A} \in \mathcal{F}_n(\lambda, \Lambda)$ and $w \in A_2$. Suppose $u \in \mathcal{D}(L^\sigma)$ is a function on \mathbb{R}^n such that $(L_w)^\sigma u = 0$ in a domain Ω , and suppose that for some $x_0 \in \Omega$, $u = 0$ on $B_1(x_0) \setminus \Omega$ where $\partial\Omega \cap B_1(x_0)$ is given by a Lipschitz graph with constant less than 1. Then there exist constants $M > 0$ and $0 < \alpha < 1$ depending on $\lambda, \Lambda, [w]_{A_2}$, and σ such that*

for all $0 < \rho < \frac{1}{2}$

$$\sup_{\Omega \cap B_\rho(x_0)} u - \inf_{\Omega \cap B_\rho(x_0)} u \leq M \left(\frac{1}{w(B_{\frac{1}{2}}(x_0))} \int_{B_{\frac{1}{2}}(x_0)} u^2 dw \right)^{\frac{1}{2}} \rho^\alpha.$$

The second type of operators we consider illustrates that the reach of our extension theorem. For subelliptic operators controlled by certain diagonal matrices, we establish a Harnack estimate for nonnegative solutions to the square root of such operators. The innovation of this application lays on the non-isotropic nature of the operators, for which the eigenvalues are allowed to vanish to different finite orders. The set where an eigenvalue vanishes may have codimension as small as one.

Remark 1.3. In the subelliptic case we only treat the square root operator $L^{1/2}$. When $0 < \sigma \neq \frac{1}{2} < 1$ the resulting equation (2.1) becomes *weighted subelliptic* with an A_2 weight depending on the new variable. We conjecture that the theory developed by Sawyer and Wheeden in [25] for subelliptic operators may be extended to include weighted subelliptic operators with A_2 weights, and, in such case, the conclusions of Theorem 1.4 would hold for all powers $0 < \sigma < 1$.

The geometry for which the Harnack's estimates hold is determined by the operator's principal terms. We now call up some relevant definitions. A vector field $X = \mathbf{v}(x) \cdot \nabla$ defined in an open set $\Omega \subset \mathbb{R}^n$ is said to be *subunit* with respect to a nonnegative quadratic form \mathcal{Q} in Ω if

$$(\mathbf{v}(x) \cdot \xi)^2 \leq \mathcal{Q}(x, \xi) \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^n.$$

Given a nonnegative matrix $\mathbf{B}(x)$, or a system of vector fields

$$\mathbb{X} = \{X_i = \mathbf{v}^i \cdot \nabla\}_{i=1}^m$$

in Ω , they determine quadratic forms $\mathcal{Q}_{\mathbf{B}}(x, \xi) = \xi \cdot \mathbf{B}(x) \xi$ and $\mathcal{Q}_{\mathbb{X}}(x, \xi) = \sum_{i=1}^m (\mathbf{v}^i \cdot \xi)^2$; a vector field X is said to be subunit with respect to either \mathbf{B} or \mathbb{X} if it is subunit with respect to the corresponding quadratic form. A Lipschitz curve $\gamma(t)$ in Ω is said to be subunit with respect to \mathcal{Q} if $\gamma'(t)$ is a subunit vector field with respect to \mathcal{Q} . Given a quadratic form \mathcal{Q} in Ω , the subunit metric associated to \mathcal{Q} is given by

$$\delta(x, y) = \inf \{r > 0 : \gamma(0) = x, \gamma(r) = y, \gamma \text{ is Lipschitz and subunit}\}.$$

This metric was introduced by Fefferman and Phong in [11] where they characterize subellipticity for operators with smooth coefficients.

The following theorem is a special case of a more general result proven in Section 3.2, in which drift terms and zero order terms are considered. We present this simplified version first for clarity. The Harnack's inequality for

subelliptic operators with rough coefficients utilized here was established in [25].

Theorem 1.4. *Let Ω be an open set in \mathbb{R}^n and let a_1, \dots, a_n be nonnegative Lipschitz functions in Ω such that for each $x_0 \in \Omega$ there exists a neighbourhood \mathcal{N} of x_0 in Ω and a permutation $\tau = \tau_{x_0}$ of the set $\{1, \dots, n\}$ so that for $\tau(y) := (y_1, \dots, y_n) = (x_{\tau(1)}, \dots, x_{\tau(n)})$ and $\tilde{a}_j(y) = a_{\tau(j)}(\tau^{-1}(y))$, we have that $\tilde{a}_1 \approx 1$ in \mathcal{N} , and*

$$\tilde{a}_j(y) = \tilde{a}_j(y_1, \dots, y_{j-1}) \approx (y_1^2 + \dots + y_{j-1}^2)^{\frac{k_j}{2}} \quad \text{in } \mathcal{N}$$

for some nonnegative integers k_j , $j = 2, \dots, n$. Let $\mathbf{B}(x)$ be an $n \times n$ measurable matrix function in Ω such that for some constants $0 < c_{\mathbf{B}} \leq C_{\mathbf{B}} < \infty$

$$c_{\mathbf{B}} \sum_{j=1}^n a_j^2(x) \xi_j^2 \leq \xi' \mathbf{B}(x) \xi \leq C_{\mathbf{B}} \sum_{j=1}^n a_j^2(x) \xi_j^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Let $\mathcal{L} = -\operatorname{div} \mathbf{B}(x) \nabla$ in Ω , suppose that $u \in \mathcal{D}(\mathcal{L}^{\frac{1}{2}})$, and that $\mathcal{L}^{\frac{1}{2}}u = 0$ in an open set $\Omega' \Subset \Omega$. Then there exist a constant $C_H > 0$ such that for every ball $B_{2r}(x_0) \subset \Omega'$, u satisfies

$$\operatorname{ess\,sup}_{B_r(x_0)} u \leq C_H \left(\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |u|^2 dx \right)^{\frac{1}{2}}.$$

where the balls B_r are the subunit metric balls of the metric induced by the system of vector fields $\left\{ a_j(x) \frac{\partial}{\partial x_j} \right\}_{j=1}^n$. Moreover, if u is nonnegative, then the Harnack's estimate holds:

$$\operatorname{ess\,sup}_{B_r} u \leq C_H \operatorname{ess\,inf}_{B_r} u.$$

Some specific examples of operators included in Theorem 1.4 are: $L_2 = \frac{\partial}{\partial x^2} + |x|^{k_1} \frac{\partial}{\partial y^2}$, and $L_3 = \frac{\partial}{\partial x^2} + |x|^{k_2} \frac{\partial}{\partial y^2} + \left(|x|^{k_3} + |y|^{k_4} \right) \frac{\partial}{\partial z^2}$, where $k_1, k_2 \geq 1$.

Our final application is in the nondivergence case for operator with coefficients with minimal regularity. We obtain a priori estimates for solutions of $\mathfrak{L}_{\mathbf{A}}^\sigma u = 0$ for σ in a range depending on p , we show that such solutions are in $C^{1,\alpha}$ for all $0 < \alpha < 1$.

Given $0 < \lambda \leq \Lambda < \infty$, and $\mathbf{A} \in \mathcal{F}_n(\lambda, \Lambda)$ we denote by $\mathfrak{L}_{\mathbf{A}}$ the nondivergence form operator

$$(1.2) \quad \mathfrak{L}_{\mathbf{A}} u = -a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = -\operatorname{trace}(\mathbf{A} D^2 u),$$

where we adopt the Einstein summation convention. The operator $\mathfrak{L}_{\mathbf{A}}$ acts on $W^{2,p}(\mathbb{R}^n)$ $1 \leq p \leq \infty$, i.e. $\mathfrak{L}_{\mathbf{A}} : W^{2,p}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$. We define the angle ω by

$$(1.3) \quad \omega = \sup_{x \in \mathbb{R}^n} \left\{ \arg(a^{ij}(x) \eta_i \bar{\eta}_j) : \eta \in \mathbb{C}^n \right\} = \arctan \left(\frac{\Lambda}{\lambda} \right) < \frac{\pi}{2}.$$

We now recall the definition of $BMO(\mathbb{R}^n)$, a measurable function f is in $BMO(\mathbb{R}^n)$ if

$$\|f\|_* = \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B| dx < \infty$$

where B ranges over all balls in \mathbb{R}^n , and $f_B = \frac{1}{|B|} \int_B f(x) dx$. For a vector or matrix function, its BMO norm is defined as the maximum of the BMO norms of each of its components.

We obtain the following a priori estimate for solutions of the fractional operator.

Theorem 1.5. *For every dimension $n \geq 1$, and constants $0 < \lambda \leq \Lambda < \infty$, there exists $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda) > 0$ such that if $\mathbf{A} \in \mathcal{F}_n(\lambda, \Lambda)$ and $\|\mathbf{A}\|_* < \varepsilon_0$ then if $0 < \sigma < \frac{p}{p+1}$ and $u \in \mathcal{D}(\mathfrak{L}_{\mathbf{A}}) \subset \mathcal{D}(\mathfrak{L}_{\mathbf{A}}^\sigma) \subset L^p(\mathbb{R}^n)$ for some $1 < p < \infty$, is such that $\mathfrak{L}_{\mathbf{A}}^\sigma u = 0$ in a nonempty open set $\Omega \subset \mathbb{R}^n$, then $u \in C^{1,\alpha}(\Omega)$ for all $0 < \alpha < 1$.*

2. THE EXTENSION THEOREM

In this section we prove the extension theorem for closed operators on a Banach space. The existence and properties of the functional calculus for closed operators (not necessarily sectorial) may be found in [14, 1].

A result similar to the following theorem was first obtained in [27] for self-adjoint second order differential operators in a Hilbert space.

Theorem 2.1. *Let \mathcal{X} be a Banach space and let T be a densely defined closed operator on $\mathcal{D}(T) \subset \mathcal{X}$. For $0 < \sigma < 1$, let $x_0 \in \mathcal{D}(T^\sigma) \subset \mathcal{X}$. The function $x : [0, \infty) \mapsto \mathcal{X}$ defined by $x(0) = x_0$ and*

$$x(y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tT} ((tT)^\sigma x_0) e^{-\frac{y^2}{4t}} \frac{dt}{t},$$

satisfies $x(y) \in \mathcal{D}(T)$ for all $y > 0$ and it is a solution of the initial value problem: $x(0) = x_0$, and

$$(2.1) \quad -Tx(y) + \frac{1-2\sigma}{y} x'(y) + x''(y) = 0 \quad \text{in } (0, \infty)$$

in the sense that $x(y) \rightarrow x_0$ in \mathcal{X} as $y \rightarrow 0^+$ and the above differential equation holds in \mathcal{X} . Moreover, we have that

$$(2.2) \quad \lim_{y \rightarrow 0^+} \frac{x(y) - x_0}{y^{2\sigma}} = \frac{\Gamma(-\sigma)}{4^\sigma \Gamma(\sigma)} T^\sigma x_0 = \frac{1}{2\sigma} \lim_{y \rightarrow 0^+} y^{1-2\sigma} x'(y);$$

$$(2.3) \quad \frac{1}{2\sigma} \lim_{y \rightarrow 0^+} y^{2-2\sigma} x''(y) = \frac{2\sigma - 1}{4^\sigma} \frac{\Gamma(-\sigma)}{\Gamma(\sigma)} T^\sigma x_0;$$

the following Poisson formula holds for x :

$$(2.4) \quad x(y) = \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-tT} e^{-\frac{y^2}{4t}} x_0 \frac{dt}{t^{1+\sigma}} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-\frac{y^2}{4r} T} x_0 \frac{dr}{r^{1-\sigma}};$$

and for all $n \geq 0$ we have the bounds

$$(2.5) \quad \left\| \frac{d}{dy^n} x(y) \right\| \leq \frac{C}{y^n} \|x_0\|, \quad \text{for all } y > 0,$$

and

$$(2.6) \quad \|T^\sigma x(y)\| \leq C \|T^\sigma x_0\|.$$

Finally, if $x_0 \in \mathcal{D}(T)$, we also have that

$$(2.7) \quad \lim_{y \rightarrow 0^+} T x(y) = T x_0.$$

Proof. First note that since $e^{-tz} (tz)^\sigma \in H_0^\infty(\Sigma_{\pi/2-\varepsilon})$ (see the Appendix for details on the functional calculus) for any fixed $0 < \varepsilon < \pi/2$, we have that $x(y) = \psi_y(T) x_0$, where

$$(2.8) \quad \psi_y(z) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tz} (tz)^\sigma e^{-\frac{y^2}{4t}} \frac{dt}{t} \in H^\infty(\Sigma_{\pi/2-\varepsilon}) \quad \text{uniformly in } y > 0$$

so by (4.6) it follows that $\|x(y)\| \leq C \|x_0\|$ where $C = \|\psi\|_{L^\infty(\Sigma_{\pi/2-\varepsilon})}$ and so $x(y)$ is well defined in \mathcal{X} . Since

$$\lim_{y \rightarrow 0^+} \psi_y(z) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tz} (tz)^\sigma \frac{dt}{t} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-s} s^\sigma \frac{ds}{s} = 1,$$

we have that $x(y) = \psi_y(T) x_0 \rightarrow x_0$ in \mathcal{X} as $y \rightarrow 0^+$. Also, since $e^{-tT} : \mathcal{X} \rightarrow \mathcal{D}(T)$ it follows that $x(y) \in \mathcal{D}(T)$ for all $y > 0$. Moreover, from the estimates

$$\left| \frac{d^m}{dy^m} e^{-\frac{y^2}{4t}} \right| \leq \frac{C_m}{y^m} \quad m = 0, 1, 2, \dots$$

with C_m independent of t , it follows that derivatives of $x(y)$ may be computed by taking derivatives inside the integral:

$$(2.9) \quad \frac{d^n}{dy^n} x(y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tT} ((tT)^\sigma x_0) \left(\frac{d^n}{dy^n} e^{-\frac{y^2}{4t}} \right) \frac{dt}{t},$$

for all $n = 0, 1, 2, \dots$. In particular, it follows as before by the boundedness of the functional calculus (4.6) that

$$\left\| \frac{d^n}{dy^n} x(y) \right\| \leq \frac{C_n}{y^n} \|x_0\| \quad \text{for all } n \geq 0.$$

This proves (2.5). We remark that for an explicit resolution of the fractional powers T^σ we may use (4.7). Now, since

$$\begin{aligned} \frac{1-2\sigma}{y} \left(\frac{d}{dy} e^{-\frac{y^2}{4t}} \right) + \left(\frac{d^2}{dy^2} e^{-\frac{y^2}{4t}} \right) &= \left(-\frac{1-2\sigma}{y} \frac{y}{2t} - \frac{1}{2t} + \frac{y^2}{4t^2} \right) e^{-\frac{y^2}{4t}} \\ &= \left(\frac{\sigma-1}{t} + \frac{y^2}{4t^2} \right) e^{-\frac{y^2}{4t}} \end{aligned}$$

we have

$$\begin{aligned} &\frac{1-2\sigma}{y} x'(y) + x''(y) \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tT} (T^\sigma x_0) t^{\sigma-1} \left(\frac{\sigma-1}{t} + \frac{y^2}{4t^2} \right) e^{-\frac{y^2}{4t}} dt \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tT} (T^\sigma x_0) \left(\frac{d}{dt} t^{\sigma-1} e^{-\frac{y^2}{4t}} \right) dt \\ &= \frac{1}{\Gamma(\sigma)} e^{-tT} (T^\sigma x_0) t^{\sigma-1} e^{-\frac{y^2}{4t}} \Big|_{t=0}^\infty \\ &\quad + \frac{1}{\Gamma(\sigma)} \int_0^\infty \left(-\frac{d}{dt} e^{-tT} \right) (T^\sigma x_0) \left(t^{\sigma-1} e^{-\frac{y^2}{4t}} \right) dt \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty T e^{-tT} ((tT)^\sigma x_0) e^{-\frac{y^2}{4t}} \frac{dt}{t} \\ &= Tx(y). \end{aligned}$$

This proves that $x(y)$ satisfies equation (2.1). Now, since

$$\frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tz} (tz)^\sigma \frac{dt}{t} = 1, \quad z \in \Sigma_{\pi/2},$$

we can write

$$x_0 = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tT} (tT)^\sigma x_0 \frac{dt}{t},$$

and therefore

$$\begin{aligned} \lim_{y \rightarrow 0^+} \frac{x(y) - x_0}{y^{2\sigma}} &= \lim_{y \rightarrow 0^+} \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tT} ((tT)^\sigma x_0) \left(\frac{e^{-\frac{y^2}{4t}} - 1}{y^{2\sigma}} \right) \frac{dt}{t} \\ &= \lim_{y \rightarrow 0^+} \frac{1}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-tT} (T^\sigma x_0) \left(\frac{e^{-\frac{y^2}{4t}} - 1}{\left(\frac{y^2}{4t}\right)^\sigma} \right) \frac{dt}{t}. \end{aligned}$$

Performing the change of variables $s = y^2 / (4t)$,

$$\lim_{y \rightarrow 0^+} \frac{x(y) - x_0}{y^{2\sigma}} = \lim_{y \rightarrow 0^+} \frac{1}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-\frac{y^2}{4s} T} (T^\sigma x_0) \left(\frac{e^{-s} - 1}{s^\sigma} \right) \frac{ds}{s}.$$

Since for any fixed $\varepsilon > 0$ $e^{-\frac{y^2}{4s} T} \rightarrow I$ as $y \rightarrow 0^+$, strongly and uniformly on $s \in [\varepsilon, \infty)$, and $\left\| e^{-\frac{y^2}{4s} T} \right\| \leq 1$ for all $s, y > 0$, we conclude that

$$\begin{aligned} \lim_{y \rightarrow 0^+} \frac{x(y) - x_0}{y^{2\sigma}} &= \frac{1}{4^\sigma \Gamma(\sigma)} T^\sigma x_0 \int_0^\infty \left(\frac{e^{-s} - 1}{s^\sigma} \right) \frac{ds}{s} \\ &= \frac{\Gamma(-\sigma)}{4^\sigma \Gamma(\sigma)} T^\sigma x_0. \end{aligned}$$

This establishes the first equality in (2.2). Similarly, from (2.9) we have

$$\begin{aligned} \frac{1}{2\sigma} \lim_{y \rightarrow 0^+} y^{1-2\sigma} x'(y) &= -\frac{1}{2\sigma} \frac{1}{\Gamma(\sigma)} \lim_{y \rightarrow 0^+} \int_0^\infty e^{-tT} ((tT)^\sigma x_0) \frac{y^{2-2\sigma}}{2t} e^{-\frac{y^2}{4t}} \frac{dt}{t} \\ &= -\frac{1}{\sigma} \frac{1}{4^\sigma \Gamma(\sigma)} \lim_{y \rightarrow 0^+} \int_0^\infty e^{-tT} (T^\sigma x_0) \left(\frac{y^2}{4t} \right)^{1-\sigma} e^{-\frac{y^2}{4t}} \frac{dt}{t} \\ &= -\frac{1}{\sigma} \frac{1}{4^\sigma \Gamma(\sigma)} \lim_{y \rightarrow 0^+} \int_0^\infty e^{-\frac{y^2}{4s} T} (T^\sigma x_0) s^{1-\sigma} e^{-s} \frac{ds}{s} \\ &= -\frac{1}{\sigma} \frac{1}{4^\sigma \Gamma(\sigma)} T^\sigma x_0 \int_0^\infty s^{1-\sigma} e^{-s} \frac{ds}{s} \\ &= -\frac{1}{\sigma} \frac{\Gamma(1-\sigma)}{4^\sigma \Gamma(\sigma)} T^\sigma x_0 = \frac{\Gamma(-\sigma)}{4^\sigma \Gamma(\sigma)} T^\sigma x_0, \end{aligned}$$

which establishes the second equality in (2.2). In this manner we also obtain

$$\begin{aligned} \frac{1}{2\sigma} y^{2-2\sigma} x''(y) &= \frac{4^{1-\sigma}}{2\sigma \Gamma(\sigma)} \int_0^\infty e^{-\frac{y^2}{4s} T} (T^\sigma x_0) s^{2-\sigma} e^{-s} \frac{ds}{s} \\ &\quad - \frac{4^{1-\sigma}}{4\sigma \Gamma(\sigma)} \int_0^\infty e^{-\frac{y^2}{4s} T} (T^\sigma x_0) s^{1-\sigma} e^{-s} \frac{ds}{s} \end{aligned}$$

which yields (2.3) as $y \rightarrow 0^+$.

Now, from (4.4) and (4.5) we have the representation (all the expressions are valued at x_0)

$$\begin{aligned} x(y) &= \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tT} (tT)^\sigma e^{-\frac{y^2}{4t}} \frac{dt}{t} \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_{\Gamma_{\pi/2-\theta}} e^{-zT} \eta(z) dz e^{-\frac{y^2}{4t}} \frac{dt}{t} \\ &= \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_{\Gamma_{\pi/2-\theta}} e^{-zT} \frac{1}{2\pi i} \int_{\gamma_\nu(z)} e^{\zeta z} e^{-t\zeta} (t\zeta)^\sigma d\zeta dz e^{-\frac{y^2}{4t}} \frac{dt}{t}. \end{aligned}$$

Since $\zeta \in \gamma_\nu(z)$ does not vanish, and using Fubini's theorem, we can perform the change of variables $t = \frac{y^2}{4\zeta\rho}$, so $dt = -\frac{y^2}{4\zeta\rho^2}d\rho$, to obtain

$$\begin{aligned} x(y) &= \frac{1}{\Gamma(\sigma)} \int_{\Gamma_{\pi/2-\theta}} e^{-zT} \frac{1}{2\pi i} \int_{\gamma_\nu(z)} e^{\zeta z} \int_0^\infty e^{-t\zeta} (t\zeta)^\sigma e^{-\frac{y^2}{4t}} \frac{dt}{t} d\zeta dz \\ &= \frac{1}{\Gamma(\sigma)} \int_{\Gamma_{\pi/2-\theta}} e^{-zT} \frac{1}{2\pi i} \int_{\gamma_\nu(z)} e^{\zeta z} \int_{\gamma_\nu(z)} e^{-\frac{y^2}{4\rho}} \left(\frac{y^2}{4\rho}\right)^\sigma e^{-\zeta\rho} \frac{d\rho}{\rho} d\zeta dz, \end{aligned}$$

where $\overline{\gamma_\nu(z)}$ is the conjugate path to $\gamma_\nu(z)$. By homotopy and the Cauchy's integral formula, the path of integration $\gamma_\nu(z)$ may be replaced by \mathbb{R}^+ , so it follows that

$$\begin{aligned} x(y) &= \frac{1}{\Gamma(\sigma)} \int_{\Gamma_{\pi/2-\theta}} e^{-zT} \frac{1}{2\pi i} \int_{\gamma_\nu(z)} e^{\zeta z} \int_0^\infty e^{-\frac{y^2}{4t}} \left(\frac{y^2}{4t}\right)^\sigma e^{-\zeta t} \frac{dt}{t} d\zeta dz \\ &= \frac{1}{\Gamma(\sigma)} \left(\frac{y^2}{4}\right)^\sigma \int_0^\infty \left(\int_{\Gamma_{\pi/2-\theta}} e^{-zT} \frac{1}{2\pi i} \int_{\gamma_\nu(z)} e^{\zeta z} e^{-\zeta t} d\zeta dz \right) e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}} \\ &= \frac{1}{\Gamma(\sigma)} \left(\frac{y^2}{4}\right)^\sigma \int_0^\infty e^{-tT} e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}} \end{aligned}$$

where we again used (4.4) and (4.5). This proves the first equality in (2.4). The second equality in (2.4) follows upon implementing the change of variables $t = \frac{y^2}{4s}$ on the second term.

Finally, from the identity $x(y) = \psi_y(T) x_0$ we have that $Tx(y) = \varphi_y(T) x_0$ with $\varphi_y(z) = z\psi_y(z)$. Now, $\varphi_y(z) \rightarrow z$ locally uniformly in $\Sigma_{\pi/2-\varepsilon}$, so if $x_0 \in \mathcal{D}(T)$ then (2.7) follows. \square

Remark 2.2. The solution $\psi(T) x_0$ with $\psi(z)$ given by (2.8) must be understood in a limit sense in the Banach space \mathcal{X} . Note that $\psi(0) = 1$, so ψ does not belong to $H_0^\infty(\Sigma_{\pi/2-\varepsilon})$. Indeed, the change of variables $s = tz$ yields

$$\psi(z) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-s} s^\sigma e^{-\frac{y^2 z}{4s}} \frac{ds}{s}$$

from which we can easily see that $\psi(z) \rightarrow 1$ as $z \rightarrow 0$ in $\Sigma_{\pi/2-\varepsilon}$ (and that $\psi(z) \rightarrow 0$ as $z \rightarrow \infty$ in $\Sigma_{\pi/2-\varepsilon}$). Thus, $\psi(T)$ shall be understood as a limit of $\psi_n(T)$ where $\psi_n \in H_0^\infty(\Sigma_{\pi/2-\varepsilon})$ and $\psi_n \rightarrow \psi$ uniformly on compact subsets of $\Sigma_{\pi/2-\varepsilon}$. For example, it would suffice to take $\psi_n(z) = \psi(z) - e^{-nz}$.

Remark 2.3. Note that the first Poisson formula in (2.4) says that the solution $x(y)$ is $x(y) = \Psi_y(T) x_0$ where the operator $\Psi_y(T)$ is given by

$$\Psi_y(T) = \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-tT} e^{-\frac{y^2}{4t}} \frac{dt}{t^{1+\sigma}}.$$

This operator is indeed the Laplace transform $\mathcal{L}(g_y)(z)$ of the function $g_y(t) = \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} t^{-(1+\sigma)} e^{-\frac{y^2}{4t}} \in L^1([0, \infty))$. The mapping $g \rightarrow \mathcal{L}(g)$ is called the *Phillips calculus* for T (see 3.3 in [14]).

3. APPLICATIONS TO DIFFERENTIAL OPERATORS

In this section we provide some applications of the general extension Theorem 2.1 to three different types of differential operators to illustrate its versatility.

3.1. Non-symmetric weighted elliptic operators. Let Ω be an open subset of \mathbb{R}^n . For indexes $1 \leq p \leq \infty$, and a locally integrable nonnegative function w we denote by $L^p(\Omega, w)$ the space of measurable functions f on Ω such that $|f|^p$ is integrable with respect to the measure $dw = w(x) dx$. Given a weight $w \in A_2(\Omega)$, the weighted Sobolev spaces $\mathcal{H}^1(\Omega, w)$ consists of those $L^2(\Omega, w)$ functions f such that $|\nabla f| \in L^2(\Omega, w)$. This space is a Hilbert space with inner product

$$(3.1) \quad \langle u, v \rangle_{\mathcal{H}^1(\Omega, w)} = \int_{\Omega} uv \, dw + \int_{\Omega} \nabla u \cdot \nabla v \, dw.$$

See [18, 10] for more details about these weighted Sobolev spaces. We also adopt the conventions $L^2(w) = L^2(\mathbb{R}^n, w)$ and $\mathcal{H}^1(w) = \mathcal{H}^1(\mathbb{R}^n, w)$. In what follows we will take $\Omega = \mathbb{R}^n$ for simplicity, but all the definitions and properties below can be specialized to any subdomain $\Omega \subset \mathbb{R}^n$.

Recall that given $0 < \lambda \leq \Lambda < \infty$, $\mathcal{F}_n(\lambda, \Lambda)$ denotes the set of real valued $n \times n$ matrices whose eigenvalues lie in the interval $[\lambda, \Lambda]$. Given any $\mathbf{A} \in \mathcal{F}_n(\lambda, \Lambda)$ we can more define a bilinear form $\mathcal{E}_{\mathbf{A}, w}$ on $\mathcal{H}^1(w) \subset L^2(w)$ by

$$\mathcal{E}_{\mathbf{A}, w}(u, v) = \langle \mathbf{A}(x) \nabla u, \nabla v \rangle_w = \int_{\mathbb{R}^n} \mathbf{A}(x) \nabla u(x) \cdot \nabla v(x) \, dw.$$

We will check that $\mathcal{E}_{\mathbf{A}, w}$ satisfies conditions (I), (II), and (III) from Definition 4.1. By ellipticity (1.1) it is clear that $\mathcal{E}_{\mathbf{A}, w}$ is nonnegative, and, moreover, $\mathbf{A}(x) \xi \cdot \xi$ is an inner product in \mathbb{R}^n , hence by Cauchy-Schwarz $|\mathcal{E}_{\mathbf{A}, w}(u, v)| \leq \mathcal{E}_{\mathbf{A}, w}(u, u)^{1/2} \mathcal{E}_{\mathbf{A}, w}(v, v)^{1/2}$ for all $u, v \in \mathcal{H}^1(w)$. Thus $\mathcal{E}_{\mathbf{A}, w}$ satisfies conditions (I) and (II). On the other hand, since the domain of $\mathcal{E}_{\mathbf{A}, w}$ is $\mathcal{H}^1(w)$, a Hilbert space with inner product (3.1), which is equivalent to

$$\mathcal{E}_{(\mathbf{A}, w), \alpha}^{(s)}(u, v) = \frac{1}{2} \left(\mathcal{E}_{(\mathbf{A}, w)}^{(s)}(u, v) + \mathcal{E}_{(\mathbf{A}, w)}^{(s)}(v, u) \right) + \alpha \langle u, v \rangle_w$$

for all $\alpha > 0$, we have that $\mathcal{E}_{\mathbf{A}, w}$ also satisfies (III). Hence, $\mathcal{E}_{\mathbf{A}, w}$ is a nonnegative bilinear form satisfying the conditions of Theorem 4.2, the associated operator $L_w = L_{\mathbf{A}, w}$ is closed and it has domain $\mathcal{D}(L_w)$ which is dense

in $L^2(w)$, and for all $u \in \mathcal{D}(L_w)$ we have that $f(x) = L_w u(x) \in L^2(w)$ satisfies

$$\int_{\mathbb{R}^n} \mathbf{A}(x) \nabla u(x) \cdot \nabla \varphi(x) dw = \int_{\mathbb{R}^n} f(x) \varphi(x) dw, \quad \text{for all } \varphi \in \mathcal{H}^1(w).$$

It is in this weak sense (integrating by parts) that we say that $L_w u(x) = -\frac{1}{w} \operatorname{div} w \mathbf{A}(x) \nabla u(x) = f(x)$. Notice that Theorem 4.2 also yields the adjoint operator \widehat{L}_w is closed with dense domain $\mathcal{D}(\widehat{L}_w) \subset L^2(w)$ such that

$$\int_{\mathbb{R}^n} \mathbf{A}(x) \nabla \varphi(x) \cdot \nabla v(x) dw = \int_{\mathbb{R}^n} \varphi(x) \widehat{L}_w v(x) dw,$$

for all $v \in \mathcal{D}(\widehat{L}_w)$, $\varphi \in \mathcal{H}^1(w)$. Moreover, by Corollary 4.6 it follows that both L_w and \widehat{L}_w are sectorial of angle $\pi/2 - \arctan(1/K)$ and they generate strongly continuous semigroups in $[0, \infty)$ and a contractive holomorphic semigroups in $\Sigma_{\arctan(1/K)}$.

Fabes, Kenig, and Serapioni extended the De Giorgi-Nash-Moser theory to symmetric weighted elliptic operators L_w with A_2 weights (and with quasi-conformal weights) [10]. As observed in their paper, the Moser iteration scheme can more generally be implemented as far as proper versions of the following a-priori estimates are available:

- (1) A Caccioppoli inequality. For every $u \in \mathcal{H}^1(w)$ and $C_0^\infty(\mathbb{R}^n)$ function φ

$$\int \varphi^2 |\nabla u|^2 dw \leq C \int |u(x)|^2 (|\nabla \varphi|^2 + \varphi^2) dw + C \int |L_w u|^2 \varphi^2 dw,$$

where $C = C(\lambda, \Lambda)$.

- (2) A Sobolev inequality. There exist $p \in [1, \infty)$, $k > 1$, and $C = C(\lambda, \Lambda, w) > 0$ such that for all balls $B_r = B_r(x) \subset \mathbb{R}^n$ and any $u \in C_0^\infty(B_r)$

$$\left(\frac{1}{w(B_r)} \int_{B_r} |u|^{pk} dw \right)^{\frac{1}{pk}} \leq Cr \left(\frac{1}{w(B_r)} \int_{B_r} |\nabla u|^p dw \right)^{\frac{1}{p}}.$$

- (3) A Poincaré inequality. There exist $p \in [1, \infty)$, $k > 1$, and $C = C(\lambda, \Lambda, w) > 0$ such that for all balls $B_r = B_r(x) \subset \mathbb{R}^n$ and any Lipschitz function u

$$\left(\frac{1}{w(B_r)} \int_{B_r} |u - u_{B_r}|^{pk} dw \right)^{\frac{1}{pk}} \leq Cr \left(\frac{1}{w(B_r)} \int_{B_r} |\nabla u|^p dw \right)^{\frac{1}{p}},$$

where $u_{B_r} = \frac{1}{w(B_r)} \int_{B_r} u dw$.

The De Giorgi-Nash-Moser techniques have been applied in increasing generality to degenerate elliptic equations, semi-linear equations and fully nonlinear equations. In most applications, Caccioppoli estimates are an easy consequence of the definition of weak solutions and integration by parts. Sobolev and Poincaré inequalities have been established in a wide variety of settings, including those of weighted elliptic operators with symmetric coefficients [10] and subelliptic equations [25, 24]. As pointed in [16], it was first observed by Morrey [20] (chapter 5) that the De Giorgi-Nash-Moser theory also holds for solutions to elliptic divergence form equations without the assumption that the matrix $\mathbf{A}(x)$ is symmetric, this fact easily extends to the weighted elliptic operators defined above. In particular, we have that local boundedness and Harnack's inequality hold in this setting.

Theorem 3.1 (Boundedness of solutions and Harnack's inequality for weighted elliptic operators). *Let $w \in A_2$ and let $\Omega \subset \mathbb{R}^n$ open. If $u \in \mathcal{H}^1(\Omega, w)$ is a solution of $L_w u = 0$ in Ω , then there exists a constant $M = M(\lambda, \Lambda, [w]_{A_2}) > 0$ such that for every ball $B_{2r}(y) \subset \Omega$*

$$\max_{B_r(y)} |u(x)| \leq M \frac{1}{w(B_{2r}(y))} \int_{B_{2r}(y)} |u|^2 dw(x).$$

Moreover, if u is nonnegative then for every ball $B_{2r}(y) \subset \Omega$

$$\max_{B_r(y)} u \leq M \min_{B_r(y)} u.$$

Note that the case $w \equiv 1$ in the above theorem covers classical non-symmetric elliptic operators. Now we are ready to apply the main extension theorem to weighted elliptic operators.

Proof of Theorem 1.1. We have $u \in \mathcal{D}((L_w)^\sigma) \subset L^2(\Omega, w)$ is a solution of $(L_w)^\sigma u = 0$ in an open set $\Omega \subset \mathbb{R}^n$. For $(x, y) \in \Omega \times (0, \infty)$ we let

$$U(x, y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tL_w} ((tL_w)^\sigma u(x)) e^{-\frac{y^2}{4t}} \frac{dt}{t}.$$

By Theorem 2.1, with $\mathcal{X} = L^2(\Omega, w)$, $U(x, y)$ is well defined and $\|U(\cdot, y)\|_{L^2(w, \Omega)} \leq C \|u\|_{L^2(w, \Omega)}$ for all $y > 0$. Moreover, U is a solution of the initial value problem

$$(3.2) \quad \begin{aligned} -L_w U(x, y) + \frac{1-2\sigma}{y} \frac{\partial}{\partial y} U(x, y) + \frac{\partial^2}{\partial y^2} U(x, y) &= 0 \\ U(x, 0) &= u(x). \end{aligned}$$

Note that the differential equation above can be written as

$$(3.3) \quad \mathcal{L}_{w, \sigma} U := -\frac{1}{\mathbf{w}(x, y)} \operatorname{div} \mathbf{w}(x, y) \mathcal{A} U(x, y) = 0$$

where, for all $x \in \Omega$ and $y \in \mathbb{R}$

$$\mathcal{A}(x, y) = \begin{pmatrix} \mathbf{A}(x) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w}(x, y) = w(x) |y|^{1-2\sigma}.$$

This implies that $U(x, y)$ is a local solution of $\mathcal{L}_{w, \sigma} U = 0$ in $\Omega \times (0, \infty)$, i.e. for all $\varphi \in \mathcal{H}_0^1(\mathbf{w}, \Omega \times (0, \infty))$ we have that

$$(3.4) \quad \int_{\Omega \times (0, \infty)} \mathcal{A}(x, y) \nabla U(x, y) \cdot \nabla \varphi(x, y) \, d\mathbf{w} = 0.$$

We now extend $U(x, y)$ for negative values of y in an even way and call this extension \tilde{U} , i.e.

$$\tilde{U}(x, y) = \begin{cases} U(x, y) & y \geq 0 \\ U(x, -y) & y < 0 \end{cases}.$$

We claim that $\tilde{U}(x, y)$ is a local weak solution of $\mathcal{L}_{w, \sigma} \tilde{U} = 0$ in $\Omega \times \mathbb{R}$. Indeed, this follows the same way as in [3]. Let $\varphi(x, y) \in \mathcal{C}_0^\infty(\Omega \times \mathbb{R})$ (note that $\mathcal{C}_0^\infty(\Omega \times \mathbb{R})$ is dense in $\mathcal{H}_0^1(\mathbf{w}, \Omega \times \mathbb{R})$) and let $\eta(y) = \eta(|y|)$ be a smooth even cutoff function such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ for $|y| \leq 1$, $\eta \equiv 0$ for $|y| \geq 2$ and $|\eta'| \leq 2$. Set also $\eta_\varepsilon(y) = \eta(\frac{y}{\varepsilon})$ for all $\varepsilon > 0$. Then by (3.4) and (3.2) it follows that

$$\begin{aligned} & \int_{\Omega \times \mathbb{R}} \mathcal{A}(x, y) \nabla \tilde{U}(x, y) \cdot \nabla \varphi(x, y) \, d\mathbf{w} \\ &= \int_{\Omega \times \mathbb{R}} \mathcal{A}(x, y) \nabla \tilde{U}(x, y) \cdot \nabla \eta_\varepsilon(y) \varphi(x, y) \, d\mathbf{w} \\ &= \int_{\varepsilon \leq |y| \leq 2\varepsilon} |y|^{1-2\sigma} \eta_\varepsilon(y) \int_{\Omega} \mathbf{A}(x) \nabla_x \tilde{U}(x, y) \cdot \nabla_x \varphi(x, y) \, dw \, dy \\ & \quad + \int_{\varepsilon \leq |y| \leq 2\varepsilon} \int_{\Omega} |y|^{1-2\sigma} \tilde{U}_y(x, y) \cdot \frac{\partial}{\partial y} (\eta_\varepsilon(y) \varphi(x, y)) \, dw \, dy \\ &= \int_{\varepsilon \leq |y| \leq 2\varepsilon} |y|^{1-2\sigma} \eta_\varepsilon(y) \int_{\Omega} \left(\frac{1-2\sigma}{y} \tilde{U}_y(x, y) + U_{yy}(x, y) \right) \varphi(x, y) \, dw \, dy \\ & \quad + \int_{\varepsilon \leq |y| \leq 2\varepsilon} \int_{\Omega} |y|^{1-2\sigma} \tilde{U}_y(x, y) \cdot \frac{\partial}{\partial y} (\eta_\varepsilon(y) \varphi(x, y)) \, dw \, dy. \end{aligned}$$

We split the last two integrals into two, depending on the sign of the integrand y . When $y > 0$, by integration by parts have

$$\begin{aligned} & \int_{\varepsilon}^{2\varepsilon} y^{1-2\sigma} \eta_\varepsilon(y) \int_{\Omega} \left(\frac{1-2\sigma}{y} \tilde{U}_y(x, y) + \frac{\partial^2}{\partial y^2} \tilde{U}(x, y) \right) \varphi(x, y) \, dw \, dy \\ & \quad + \int_{\varepsilon}^{2\varepsilon} \int_{\Omega} y^{1-2\sigma} \tilde{U}_y(x, y) \cdot \frac{\partial}{\partial y} (\eta_\varepsilon(y) \varphi(x, y)) \, dw \, dy \\ &= \int_{\Omega} \int_{\varepsilon}^{2\varepsilon} \eta_\varepsilon(y) \frac{\partial}{\partial y} (y^{1-2\sigma} U_y(x, y)) \varphi(x, y) \, dy \, dw \end{aligned}$$

$$\begin{aligned}
& + \int_{\varepsilon}^{2\varepsilon} \int_{\Omega} y^{1-2\sigma} U_y(x, y) \cdot \frac{\partial}{\partial y} (\eta_{\varepsilon}(y) \varphi(x, y)) \, dw \, dy \\
& = \int_{\Omega} \eta_{\varepsilon}(y) (y^{1-2\sigma} U_y(x, y)) \varphi(x, y) \Big|_{y=\varepsilon}^{y=2\varepsilon} \, dw \\
& = \int_{\Omega} \varepsilon^{1-2\sigma} U_y(x, \varepsilon) \varphi(x, \varepsilon) \, dw.
\end{aligned}$$

A similar treatment for the region where $y < 0$ yields

$$\begin{aligned}
& \int_{\Omega \times \mathbb{R}} \mathcal{A}(x, y) \nabla \tilde{U}(x, y) \cdot \nabla \varphi(x, y) \, d\mathbf{w} \\
& = \int_{\Omega} \varepsilon^{1-2\sigma} U_y(x, \varepsilon) (\varphi(x, \varepsilon) + \varphi(x, -\varepsilon)) \, dw.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and $\varepsilon^{1-2\sigma} \frac{\partial U}{\partial y}(x, \varepsilon) \rightarrow (L_w)^{\sigma} u = 0$ by (2.2), we have that

$$\int_{\Omega \times \mathbb{R}} \mathcal{A}(x, y) \nabla \tilde{U}(x, y) \cdot \nabla \varphi(x, y) \, d\mathbf{w} = 0$$

which show that $\tilde{U}(x, y)$ is a local weak solution of $\mathcal{L}_{w, \sigma} \tilde{U} = 0$ in $\Omega \times \mathbb{R}$. We note that $w \in A_2(\mathbb{R}^n) \implies \mathbf{w} = w(x) |y|^{1-2\sigma} \in A_2(\mathbb{R}^{n+1})$ for all $0 < \sigma < 1$. Indeed, for fixed $(x_0, y_0) \in \mathbb{R}^{n+1}$ and $r > 0$

$$\begin{aligned}
& \left(\frac{1}{|B_r(x_0, y_0)|} \int_{B_r(x_0, y_0)} \mathbf{w}(x, y) \, dxdy \right)^{\frac{1}{2}} \left(\frac{1}{|B_r(x_0, y_0)|} \int_{B_r(x_0, y_0)} \frac{1}{\mathbf{w}(x, y)} \, dxdy \right)^{\frac{1}{2}} \\
& \leq C_n \left(\left(\frac{1}{|B_r(x_0)|} \int_{|x-x_0|<r} w(x) \, dx \right) \left(\frac{1}{|B_r(x_0)|} \int_{|x-x_0|<r} \frac{1}{w(x)} \, dx \right) \right)^{\frac{1}{2}} \\
& \quad \times \left(\left(\frac{1}{2r} \int_{|y-y_0|<r} |y|^{1-2\sigma} \, dy \right) \left(\frac{1}{2r} \int_{|y-y_0|<r} |y|^{2\sigma-1} \, dy \right) \right)^{\frac{1}{2}} \\
& \leq C_n [w]_{A_2(\mathbb{R}^n)} [|y|^{2\sigma-1}]_{A_2(\mathbb{R})} < \infty.
\end{aligned}$$

By Theorem 3.1 it follows that for all $x_0 \in \Omega$ and $r > 0$ such that $B_{2r}(x_0) \subset \Omega$

$$\begin{aligned}
(3.5) \quad \max_{B_r(x_0)} |u(x)| & \leq \max_{B_r(x_0, 0)} \left| \tilde{U}(x, y) \right| \\
& \leq M \frac{1}{\mathbf{w}(B_{2r}(x_0, 0))} \int_{B_{2r}(x_0, 0)} \left| \tilde{U}(x, y) \right|^2 d\mathbf{w}(x, y).
\end{aligned}$$

Since any A_2 weight is doubling, there exists a constant $D_w > 1$ such that

$$\mathbf{w}(B_{2r}(x_0, 0)) \geq D_w^{-1} \mathbf{w}(B_{2r}(x_0) \times [-2r, 2r])$$

$$\begin{aligned}
&= D_w^{-1} \int_{-2r}^{2r} |y|^{1-2\sigma} \int_{B_{2r}(x_0)} w(x) dx dy \\
&= D_w^{-1} \frac{(2r)^{2-2\sigma}}{1-\sigma} w(B_{2r}(x_0)).
\end{aligned}$$

We also have that

$$\begin{aligned}
\int_{B_{2r}(x_0,0)} \left| \tilde{U}(x,y) \right|^2 d\mathbf{w}(x,y) &\leq \int_{-2r}^{2r} |y|^{1-2\sigma} \int_{B_{2r}(x_0)} \left| \tilde{U}(x,y) \right|^2 w(x) dx dy \\
&= 2 \int_0^{2r} y^{1-2\sigma} \int_{B_{2r}(x_0)} |U(x,y)|^2 w(x) dx dy \\
&\leq 2 \int_0^{2r} y^{1-2\sigma} \|U(\cdot, y)\|_{L^2(w,\Omega)}^2 dy \\
(3.6) \quad &\leq C \frac{(2r)^{2-2\sigma}}{1-\sigma} \|u\|_{L^2(w,\Omega)}^2.
\end{aligned}$$

Putting these inequalities together with (3.5) yields

$$\max_{B_r(x_0)} |u(x)| \leq \frac{CMD_w}{w(B_{2r}(x_0))} \|u\|_{L^2(w,\Omega)}^2.$$

Which shows that u is locally bounded in Ω .

On the other hand we note that since the semigroup e^{-tL_w} is positive (cf. Section 4.5 in [22]), hence if u is moreover nonnegative it follows that \tilde{U} is nonnegative. Then, by Theorem 3.1

$$\max_{B_r(x_0)} u(x) \leq \max_{B_r(x_0,0)} \tilde{U}(x,y) \leq M \min_{B_r(x_0,0)} \tilde{U}(x,y) \leq M \min_{B_r(x_0)} u(x).$$

This proves the Harnack's estimate for nonnegative solutions of $(L_w)^\sigma u = 0$. Hölder's continuity of solutions follows directly from these scale invariant estimates. \square

The proof of the boundary Harnack principle for fractional powers of L_w (Theorem 1.2) is just the same as the proof of Theorem 5.3 in [3], so we only provide a sketch.

Proof of Theorem 1.2. Let $\tilde{U}(x,y)$ be as in the previous proof, then \tilde{U} is a solution of (3.3) in $\Omega \times \mathbb{R}$. Applying Theorem 2.4.6 from [10] (boundary Hölder continuity) in the set G_1 , where

$$G_\rho = B_\rho(x_0) \times (-\rho, \rho) \bigcap \Omega \times \mathbb{R},$$

yields

$$(3.7) \quad \sup_{G_\rho} \tilde{U}(x,y) - \inf_{G_\rho} \tilde{U}(x,y) \leq M \left(\int_{G_{\frac{1}{2}}} \left| \tilde{U}(x,y) \right|^2 d\mathbf{w}(x,y) \right)^{\frac{1}{2}} \rho^\alpha.$$

Since

$$\sup_{B_\rho(x_0) \cap \Omega} u(x) - \inf_{B_\rho(x_0) \cap \Omega} u(x) \leq \sup_{G_\rho} \tilde{U}(x, y) - \inf_{G_\rho} \tilde{U}(x, y)$$

the Theorem follows by bounding the right hand side in (3.7) in a similar way as we obtained (3.6).

$$\begin{aligned} \int_{G_{\frac{1}{2}}} |\tilde{U}(x, y)|^2 d\mathbf{w}(x, y) &\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |y|^{1-2\sigma} \int_{B_{\frac{1}{2}}(x_0) \cap \Omega} |\tilde{U}(x, y)|^2 w(x) dx dy \\ &= 2 \int_0^{\frac{1}{2}} y^{1-2\sigma} \int_{B_{\frac{1}{2}}(x_0) \cap \Omega} |U(x, y)|^2 w(x) dx dy \\ &\leq 2 \int_0^{2r} y^{1-2\sigma} \|U(\cdot, y)\|_{L^2(w, \Omega)}^2 dy \\ (3.8) \quad &\leq C \frac{(2r)^{2-2\sigma}}{1-\sigma} \|u\|_{L^2(w, \Omega)}^2. \end{aligned}$$

□

Let $0 < \sigma \leq 1$, for every $\mathbf{A} \in \mathcal{F}_n(\lambda, \Lambda)$ and $w \in A_2$. Suppose $u \in \mathcal{D}(L^\sigma)$ is a function on \mathbb{R}^n such that $(L_w)^\sigma u = 0$ in a domain Ω , and suppose that for some $x_0 \in \Omega$, $u = 0$ on $B_1(x_0) \setminus \Omega$ where $\partial\Omega \cap B_1(x_0)$ is given by a Lipschitz graph with constant less than 1. Then there exist constants $M > 0$ and $0 < \alpha < 1$ depending on $\lambda, \Lambda, [w]_{A_2}$, and σ such that for all $0 < \rho < \frac{1}{2}$

$$\sup_{\Omega \cap B_\rho(x_0)} u - \inf_{\Omega \cap B_\rho(x_0)} u \leq M \left(\frac{1}{w(B_{\frac{1}{2}}(x_0))} \int_{B_{\frac{1}{2}}(x_0)} u^2 dw \right)^{\frac{1}{2}} \rho^\alpha.$$

3.2. Non isotropic operators. In [25] Sawyer and Wheeden consider the general linear second order equations

$$(3.9) \quad \mathcal{L}u = -\operatorname{div} \mathbf{B}(x) \nabla u + \sum_{i=1}^N b_i R_i u + \sum_{i=1}^N S'_i c_i u + du = f + \sum_{i=1}^N T'_i g_i,$$

for which the principal part is nonnegative but not necessarily strongly elliptic. More precisely, these authors assumed the following conditions:

- (A) \mathbf{B} is a bounded measurable nonnegative semidefinite matrix,
- (B) $\{R_i\}_{i=1}^N$, $\{S_i\}_{i=1}^N$, and $\{T_i\}_{i=1}^N$, are collections of vector fields subunit with respect to $\mathbf{B}(x)$, i.e.

$$(X(x) \cdot \xi)^2 \leq \xi' \mathbf{B}(x) \xi \quad \text{for all } \xi \in \mathbb{R}^n,$$

$X = R_i, S_i, T_i$, $i = 1, \dots, N$, and all x in a domain (open and connected set) $\Omega \subset \mathbb{R}^n$;

(C) the operator coefficients $\mathbf{b} = \{b_i\}_{i=1}^N$, $\mathbf{c} = \{c_i\}_{i=1}^N$, d , and the inhomogeneous data $\mathbf{g} = \{g_i\}_{i=1}^N$ and f are measurable.

(D) The coefficients and data moreover satisfy

$$\begin{aligned} \|d\|_{L^{\frac{q}{2}}(\Omega)} + \|\mathbf{b}\|_{L^q(\Omega)} + \|\mathbf{c}\|_{L^q(\Omega)} &: \equiv N_q < \infty \\ \|f\|_{L^{\frac{q}{2}}(\Omega)} + \|\mathbf{g}\|_{L^q(\Omega)} &: \equiv N'_q < \infty \end{aligned}$$

for some $q \geq 2$.

When ellipticity is allowed to degenerate, the concept of weak solution must be adapted to the geometry induced by the principal part of the operator, namely, the geometry of the subunit metric with respect to the matrix \mathbf{B} , as described in the introduction. The natural space for solutions is also determined by the quadratic form given by \mathbf{B} , we let $W_{\mathbf{B}}^{1,2}(\Omega)$ be the space of square integrable measurable functions f such that their gradient belongs to the space $\mathbf{L}_{\mathbf{B}}^2$ given by measurable vector functions $\mathbf{v}(x)$ such that

$$\|\mathbf{v}\|_{\mathbf{L}_{\mathbf{B}}^2}^2 = \int_{\Omega} \mathbf{v} \cdot \mathbf{B} \mathbf{v} \, dx < \infty.$$

We say that u is a solution of (3.9) in Ω if $u \in W_{\mathbf{B}}^{1,2}(\Omega)$ and

$$\begin{aligned} & \int \nabla u \cdot \mathbf{B} \nabla w + \int \sum_{i=1}^N b_i (R_i u) w + \int \sum_{i=1}^N c_i u (S_i w) + \int duw \\ &= \int f w + \int \sum_{i=1}^N g_i (T_i w), \end{aligned}$$

for all nonnegative $w \in (W_{\mathbf{B}}^{1,2})_0(\Omega)$. We will further assume that the matrix \mathbf{B} is equivalent to a special kind of diagonal matrices (see condition (3.5) below) what will ensure that $W_{\mathbf{B}}^{1,2}(\Omega)$ is a Hilbert space with inner product given by

$$(3.10) \quad \langle f, g \rangle_{W_{\mathbf{B}}^{1,2}(\Omega)} = \int_{\Omega} f g \, dx + \int_{\Omega} \nabla f \cdot \mathbf{B} \nabla g \, dx.$$

See [26, 19] for details.

Note that the operator L is given by the bilinear form

$$(3.11) \quad \mathcal{E}(u, w) = \int \nabla u \cdot \mathbf{B} \nabla w + \int \sum_{i=1}^N b_i (R_i u) w + \int \sum_{i=1}^N c_i u (S_i w) + \int duw$$

on $\mathcal{F} = (W_{\mathbf{B}}^{1,2})(\Omega) \subset L^2(\Omega)$. We will assume this bilinear form is nonnegative; for this would suffice that \mathbf{b} , \mathbf{c} , and d vanish, or that $d \geq d_0 > 0$

for some constant d_0 , and $\|\mathbf{b}\|_\infty, \|\mathbf{c}\|_\infty$ small enough in view of the subunit condition (2) above. Indeed, for the second term in the (3.11) we have

$$\begin{aligned} \left| \int \sum_{i=1}^N b_i (R_i u) u \right| &\leq \sum_{i=1}^N \left(\int |R_i u|^2 \right)^{\frac{1}{2}} \left(\int b_i^2 u^2 \right)^{\frac{1}{2}} \\ &\leq N \|\mathbf{b}\|_\infty \left(\int \nabla u \mathbf{B} \nabla u \right)^{\frac{1}{2}} \left(\int u^2 \right)^{\frac{1}{2}}, \end{aligned}$$

with a similar estimate for the third term in (3.11); thus,

$$\begin{aligned} (3.12) \quad \mathcal{E}(u, u) &\geq \int \nabla u \mathbf{B} \nabla u \\ &\quad + N (\|\mathbf{b}\|_\infty + \|\mathbf{c}\|_\infty) \left(\int \nabla u \mathbf{B} \nabla u \right)^{\frac{1}{2}} \left(\int u^2 \right)^{\frac{1}{2}} + \int du^2. \end{aligned}$$

It easily follows that the right hand side is nonnegative under the above assumptions. Hence, \mathcal{E} satisfies the lower bound (I) in Definition 4.1, while the sector condition (II) can similarly be verified. We will assume further structural assumptions to guarantee a-priori Harnack estimated for solutions. First we must include some more definitions and background.

Definition 3.2 (Reverse Hölder infinity). A nonnegative function $a(t)$ defined on an open subset J of \mathbb{R} satisfies the reverse Hölder condition of infinite order if

$$\operatorname{ess\,sup}_I a(t) \leq C \frac{1}{|I|} \int_I a(t) \, dt$$

for all intervals $I \subset J$. In such case, we say that $a \in RH^\infty(J)$.

Remark 3.3. All positive powers are in RH^∞ .

Definition 3.4 (Flag condition - [25] Definition 12). A collection of continuous vector fields satisfies the flag condition at $x \in \Omega$ if for each index set $\emptyset \subset \mathcal{I} \subsetneq \{1, 2, \dots, n\}$, there is $j \notin \mathcal{I}$ such that for any neighbourhood \mathcal{N} of x in Ω , a_j does not vanish identically on $(x + \mathcal{V}_{\mathcal{I}}) \cap \mathcal{N}$ where $\mathcal{V}_\emptyset = \{0\}$ and $\mathcal{V}_{\mathcal{I}} = \operatorname{span} \{\mathbf{e}_i : i \in \mathcal{I}\}$, $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ (with 1 in the i^{th} position). The vector fields X_i satisfy the flag condition in Ω if they satisfy the flag condition at every point $x \in \Omega$.

The flag condition ensures that the flow of the vector fields X_j does not get "trapped" into any variety of dimension less than n . This condition is necessary for subellipticity of operators given by diagonal, Lipschitz, RH^∞ , vector fields (see [25], Theorem 16).

We assume that the principal part $-\operatorname{div} \mathbf{B} \nabla$ of our operator \mathcal{L} satisfies the following structural condition:

Condition 3.5. *There exist nonnegative Lipschitz functions a_j on $\Omega \subset \mathbb{R}^n$, $j = 1, \dots, n$, such that*

- (I) *for every compact set $K \subset \Omega$, $a_j \in RH^\infty(K)$ in each variable x_i with $i \neq j$, uniformly in the remaining variables;*
- (II) *the set $\mathbb{X} = \{X_j\}_{j=1}^n$ of vector fields $X_j = a_j \frac{\partial}{\partial x_j}$ satisfies the flag condition in Ω ;*
- (III) *there exist constants $0 < c_{\mathbf{B}} \leq C_{\mathbf{B}} < \infty$ such that the matrix $\mathbf{B}(x)$ satisfies the upper and lower bounds in Ω*

$$(3.13) \quad c_{\mathbf{B}} \sum_{j=1}^n a_j(x) \xi_j^2 \leq \xi' \mathbf{B}(x) \xi \leq C_{\mathbf{B}} \sum_{j=1}^n a_j(x) \xi_j^2.$$

In particular, this structure allows for different order of vanishing of the eigenvalues a_j of \mathbf{B} , what was not permitted to the weighted elliptic operators in [10] treated in Section 3.1.

The following is an example of a diagonal system of vector fields which satisfies the flag condition and condition 3.5.

Example 3.6. *Suppose $a_j(x)$, $j = 1, \dots, n$, are nonnegative Lipschitz functions, that for each $x_0 \in \Omega$ there exists a permutation $\tau = \tau_{x_0}$ of the set $\{1, \dots, n\}$ and a neighbourhood $\mathcal{N}_{x_0} \subset \Omega$ of x_0 such that in \mathcal{N}_{x_0} we have for $(y_1, \dots, y_n) = (x_{\tau(1)}, \dots, x_{\tau(n)})$ and $\tilde{a}_j(y) = a_{\tau(j)}(\tau^{-1}(y))$:*

- $\tilde{a}_1(y) \approx 1$, and $\tilde{a}_j(y) = \tilde{a}_j(y_1, \dots, y_{j-1})$ for $j = 2, \dots, n$;
- \tilde{a}_j has isolated zeroes in their variables; i.e.

$$Z_j = \{(y_1, \dots, y_{j-1}) : \tilde{a}_j(y_1, \dots, y_{j-1}) = 0\} \bigcap \mathcal{N}$$

is a discrete set in \mathbb{R}^{j-1} for $j = 2, \dots, n$;

- \tilde{a}_j is locally homogeneous of finite type: if $z \in Z_j$ then

$$\tilde{a}_j(w - z) \approx |w - z|^{k_j} \quad w \text{ near } z$$

for some integers $k_j \geq 1$, $2 \leq j \leq n$.

Then $X_j = a_j \frac{\partial}{\partial x_j}$, $1 \leq j \leq n$, is a collection of vector fields which satisfies the flag condition and (I) from condition 3.5.

Proof. As noted in [25] (Remark 13), to check that $\{X_j\}_{j=1}^n$ satisfies the flag condition at a point x_0 it suffices show that there exist an increasing sequence of index sets

$$\emptyset \neq \mathcal{I}_1 \subsetneq \mathcal{I}_2 \subsetneq \dots \subsetneq \mathcal{I}_n = \{1, \dots, n\},$$

such that for $\mathcal{V}_0 = \{0\}$ and $\mathcal{V}_j = \text{span}\{e_i : i \in \mathcal{I}_j\}$, a_i does not vanish identically on $(x_0 + \mathcal{V}_j) \cap \mathcal{N}$ for any neighbourhood \mathcal{N} of x_0 and any $i \in$

\mathcal{I}_{j+1} , $j = 0, 1, \dots, n-1$. It suffices to check this when the permutation τ is the identity and the point x_0 is the origin. From the definition of a_j , taking

$$\mathcal{I}_j = \{1, \dots, j\}, \quad j = 1, \dots, n,$$

we have that if $i+1 \in \mathcal{I}_j$, for some $j = 2, \dots, n$, then $1 \leq i \leq j-1$. Let 0_i be the origin in \mathbb{R}^i . If $a_i(0_i) > 0$ then there is nothing to prove since a_i is continuous. If $a_i(0_i) = 0$ then $a_i(w) \approx |w|^{k_i}$ for $w \in \mathbb{R}^i$ near 0_i and since $\mathcal{V}_j \cap \mathcal{N} \supset \mathbb{R}^i \cap \mathcal{N}$ we see that a_i does not vanish identically on $(x_0 + \mathcal{V}_j) \cap \mathcal{N}$ for any neighbourhood \mathcal{N} .

Finally, if $a_i(0_i) > 0$ then a_i is locally constant and therefore a_i satisfies (I) from condition 3.5. On the other hand, if $a_i(0_i) = 0$ and $i < \ell \leq n$ then a_i is constant in the variable w_ℓ so a_i is in RH^∞ of this variable independently of the remaining variables, while if $a_i(0_i) = 0$ and $1 \leq \ell \leq i$ then

$$a_i(w) \approx (w_1^2 + \dots + w_\ell^2 + \dots + w_i^2)^{\frac{k_i}{2}}$$

so a_i is in RH^∞ of the variable w_ℓ , uniformly on the remaining variables. \square

The following Harnack inequality can be found in [25] (Propositions 58 and 67 and Theorems 61 and 82), see also [19].

Theorem 3.7 (Harnack's inequality for subelliptic almost-diagonal operators). *Let \mathcal{L} be given by (3.9) where the coefficients satisfy (A), (B), (C), and (D), and $\mathbf{B}(x)$ satisfies condition 3.5 in a domain $\Omega \subset \mathbb{R}^n$. Then there exist a constant $C_H > 0$ such that every weak solution of $\mathcal{L}u = 0$ in $B_{2r}(y) \subset \Omega$, satisfies*

$$\operatorname{ess\,sup}_{B_r} u \leq C_H \left(\frac{1}{|B_{2r}(y)|} \int_{B_{2r}(y)} |u|^2 dx \right)^{\frac{1}{2}}.$$

where the balls B_r are the subunit metric balls of the metric induced by \mathbb{X} . Moreover, if u is nonnegative, then we also have that

$$\operatorname{ess\,sup}_{B_r} u \leq C_H \operatorname{ess\,inf}_{B_r} u.$$

Under Condition 3.5, the space $W_{\mathbf{B}}^{1,2}(\Omega)$ is a Hilbert space with inner product given by the left hand side of (3.10) (see [26], Theorem 2 and Section 3); from this it follows that the completeness condition (III) holds for the form \mathcal{E} in (3.12). Thus, \mathcal{E} satisfies the hypotheses of Corollary 4.6, and therefore L is a sectorial operator on $W_{\mathbf{B}}^{1,2}(\Omega)$. This allows us to apply the functional calculus of Section 4.2 to L .

With these preliminaries laid down, we can not prove Theorem 1.4; we will obtain this theorem as a consequence of the following more general result:

Theorem 3.8. *Let \mathcal{L} be given by (3.9) where the coefficients satisfy (A), (B), (C), and (D), the bilinear form \mathcal{E} given by (3.11) is nonnegative, and $\mathbf{B}(x)$ satisfies condition 3.5 in a domain $\Omega \subset \mathbb{R}^n$. Suppose $u \in \mathcal{D}(L^{\frac{1}{2}})$ and that $L^{\frac{1}{2}}u = 0$ in an open set $\Omega' \Subset \Omega$. Then there exist a constant $C_H > 0$ such that if $B_{2r}(x_0) \subset \Omega'$, then*

$$\operatorname{ess\,sup}_{B_r} u \leq C_H \left(\frac{1}{|B_{2r}(x_0)|} \int_{B_{2r}(x_0)} |u|^2 dx \right)^{\frac{1}{2}}.$$

where the balls B_r are the subunit metric balls of the metric induced by the system of vector fields $\left\{ a_j(x) \frac{\partial}{\partial x_j} \right\}_{j=1}^n$. Moreover, if u is nonnegative, then we also have that

$$\operatorname{ess\,sup}_{B_r} u \leq C_H \operatorname{ess\,inf}_{B_r} u.$$

In view of Example 3.6, is easy to check that the operator in Theorem 1.4 satisfies the hypotheses of Theorem 3.8.

Proof of Theorem 3.8. For $u \in \mathcal{D}(L^{\frac{1}{2}})$ in $\Omega \subset \mathbb{R}^n$, let $U(x, y)$ be the extension of u given by Theorem 2.1, i.e.

$$U(x, y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t\mathcal{L}} ((tT)^\sigma u(x)) e^{-\frac{y^2}{4t}} \frac{dt}{t}, \quad y > 0.$$

From (2.1) it readily follows that U satisfies the equation

$$-LU(x, y) + \frac{\partial}{\partial y^2} U(x, y) = 0 \quad \text{in } \Omega \times (0, \infty),$$

which can be written in the form

$$\begin{aligned} \mathcal{L}U(x, y) &= -\operatorname{div} \mathcal{B} \nabla U(x, y) + \sum_{i=1}^N b_i(x) R_i(x) U(x, y) \\ &\quad + \sum_{i=1}^N S'_i(x) c_i(x) U(x, y) + d(x) U(x, y) = 0, \end{aligned}$$

where

$$\mathcal{B}(x, y) = \mathcal{B}(x) = \begin{pmatrix} \mathbf{B}(x) & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies Condition 3.5, since \mathbf{B} does. Now, by assumption we have that $L^{\frac{1}{2}}u = 0$ in $\Omega' \Subset \Omega$. We extend U from $\Omega' \times (0, \infty)$ to $\Omega' \times \mathbb{R}$ as an even function as before

$$\tilde{U}(x, y) = \begin{cases} U(x, y) & x \in \Omega', y \geq 0 \\ U(x, -y) & x \in \Omega', y < 0 \end{cases}.$$

The proof that the extended function is a solution of $\mathcal{L}\tilde{U} = 0$ in $\Omega' \times \mathbb{R}$ is similar to the proof for weighted elliptic operators provided in Section 3.1; we point out that the crucial part of this proof is dealing with the principal term of the operator, which structurally is included in the operators considered in Section 3.1. We omit the details.

An application of Theorem 3.7 to this solution and the fact that $u(x) = U(x, 0)$ finishes the proof of Theorem 3.8. \square

3.3. Nondivergence Form Operators. For the type of operators (1.2) some extra hypotheses are required to guarantee existence and uniqueness of solutions to Dirichlet problems. In the classic text [13] it is shown that if the coefficients are uniformly continuous the Dirichlet $\mathfrak{L}_{\mathbf{A}}u = f$ problem has a unique solution in any $C^{1,1}$ bounded domain if $f \in L^p$ for $p > n/2$ and the boundary values are continuous. The most general existence and a-priori regularity results known today requires that the coefficients a^{ij} have small BMO norm [5, 23].

In [8] Duong and Yan prove the following result about the resolvent set of $\mathfrak{L}_{\mathbf{A}}$ (Lemma 3.1 in [8]):

Lemma 3.9. *Let $\mathfrak{L}_{\mathbf{A}}$ be given by (1.2) satisfy (1.1), and for ω given by (1.3) let $\theta \in (\omega, \pi]$. Then there exist positive constants ε_0 and C_θ such that if $\sup_{1 \leq i, j \leq n} \|a^{ij}\|_* < \varepsilon_0$ then $-\Sigma_{\pi-\theta} \subset \rho(\mathfrak{L}_{\mathbf{A}})$ and*

$$|z| \left\| (z - \mathfrak{L}_{\mathbf{A}})^{-1} \right\|_{\mathcal{B}(L^p(\mathbb{R}^n))} \leq C_\theta \quad \text{for all } z \in -\Sigma_{\pi-\theta}.$$

Moreover $\mathfrak{L}_{\mathbf{A}}$ is one-to-one and it has dense range.

In fact, if the BMO norm of the coefficients is small enough it follows that $\mathfrak{L}_{\mathbf{A}}$ is indeed an operator of type ω as described in Definition 4.3. It only remains to show that $\mathfrak{L}_{\mathbf{A}}$ is closed. This follows from the a-priori estimates in [4] obtained for VMO coefficients, and the fact that only small BMO norm was used in their proofs (see [5, 23]). Indeed, from these papers it follows that the a-priori estimates obtained for operators with uniformly continuous coefficients in [13] (Theorem 9.11) hold for operators with small BMO norm. More precisely:

Theorem 3.10 ([5]). *or $\mathfrak{L}_{\mathbf{A}}$ be given by (1.2) satisfy (1.1), if Ω is any open bounded set in \mathbb{R}^n and $1 \leq p < \infty$, there exists $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda, p, \Omega) > 0$ such that if $\max_{1 \leq i, j \leq n} \|a^{ij}\|_* \leq \varepsilon_0$ in Ω , and $u \in W_{\text{loc}}^{2,p}(\Omega) \cap L^p(\Omega)$, is such that $\mathfrak{L}_{\mathbf{A}}u = f \in L^p(\Omega)$, then for any $\Omega' \Subset \Omega$*

$$\|u\|_{W^{2,p}(\Omega')} \leq C \left(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \right),$$

where C depends on $n, \lambda, \Lambda, \varepsilon_0, p$, $\text{diam } \Omega'$ and $\text{dist}(\Omega', \partial\Omega)$.

The local estimates can be extended globally to all of \mathbb{R}^n if u and $f \in L^p(\mathbb{R}^n)$ and the BMO-norm of the coefficients is small enough in cubes. Indeed, it suffices to cover \mathbb{R}^n with a grid of closed unit cubes $\{Q_i\}_{i=1}^\infty$ and let \widetilde{Q}_i denote the union of Q_i with its $3^n - 1$ immediate adjacent cubes. Then for each Q_i we have that

$$\begin{aligned} \|u\|_{W^{2,p}(Q_i)}^p &= \|u\|_{L^p(Q_i)}^p + \|Du\|_{L^p(Q_i)}^p + \|D^2u\|_{L^p(Q_i)}^p \\ &\leq C^p \left(\|u\|_{L^p(\widetilde{Q}_i)}^p + \|f\|_{L^p(\widetilde{Q}_i)}^p \right) \end{aligned}$$

where $C = C(n, \lambda, \Lambda, \varepsilon_0, p)$ as in Theorem 3.10, is independent of each cube. Summing in i and using that the dilated cubes \widetilde{Q}_i have finite overlapping yields

$$(3.14) \quad \|u\|_{W^{2,p}(\mathbb{R}^n)} \leq C \left(\|u\|_{L^p(\mathbb{R}^n)} + \|\mathfrak{L}_{\mathbf{A}} u\|_{L^p(\mathbb{R}^n)} \right).$$

From this global estimate it follows that $\mathfrak{L}_{\mathbf{A}}$ is closed, and therefore $\mathfrak{L}_{\mathbf{A}} : W^{2,p}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is surjective by Lemma 3.9. This proves that under the extra hypothesis on the coefficients of having small enough BMO norm, the operator is of type ω , and hence it has a functional calculus and, in particular, the fractional powers $\mathfrak{L}_{\mathbf{A}}^\sigma$ are well defined for $0 < \sigma < 1$.

We note that the global estimates (3.14) are false in general without the small BMO norm assumption. In [7] the authors showed that for each $1 < p < \infty$ there exist an operator in \mathbb{R}^2 with constant coefficients in each quadrant such that (3.14) does not hold.

Applying the extension Theorem 2.1 to any $u \in \mathcal{D}(\mathfrak{L}_{\mathbf{A}}^\sigma)$, $0 < \sigma < 1$, we have that the function

$$(3.15) \quad U_\sigma(x, y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t\mathfrak{L}_{\mathbf{A}}} ((t\mathfrak{L}_{\mathbf{A}})^\sigma u(x)) e^{-\frac{y^2}{4t}} \frac{dt}{t}$$

satisfies $U_\sigma(\cdot, y) \in \mathcal{D}(\mathfrak{L}_{\mathbf{A}}) = W^{2,p}(\mathbb{R}^n)$ for all $y > 0$ and it is a solution of the initial value problem: $U_\sigma(x, 0) = u(x)$, and

$$(3.16) \quad -\mathfrak{L}_{\mathbf{A}} U_\sigma(x, y) + \frac{1-2\sigma}{y} \frac{\partial}{\partial y} U_\sigma(x, y) + \frac{\partial^2}{\partial y^2} U_\sigma(x, y) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty);$$

with the bounds

$$(3.17) \quad \left\| \frac{\partial^m}{\partial y^m} U_\sigma(\cdot, y) \right\|_{L^p(\mathbb{R}^n)} \leq \frac{C_m}{y^m} \|u\|_{L^p(\mathbb{R}^n)}, \quad m = 0, 1, \dots; y > 0.$$

From Theorem 2.1 we also have the estimates

$$\frac{\Gamma(-\sigma)}{4^\sigma \Gamma(\sigma)} \mathfrak{L}_{\mathbf{A}}^\sigma u = \frac{1}{2\sigma} \lim_{y \rightarrow 0^+} y^{1-2\sigma} \frac{\partial}{\partial y} U_\sigma(\cdot, y)$$

$$(3.18) \quad = \lim_{y \rightarrow 0^+} \frac{U_\sigma(\cdot, y) - u(\cdot)}{y^{2\sigma}};$$

and

$$(3.19) \quad \frac{2\sigma - 1}{4\sigma} \frac{\Gamma(-\sigma)}{\Gamma(\sigma)} \mathfrak{L}_\mathbf{A}^\sigma u = \frac{1}{2\sigma} \lim_{y \rightarrow 0^+} y^{2-2\sigma} \frac{\partial^2}{\partial y^2} U(\cdot, y).$$

where the convergence is in $L^p(\mathbb{R}^n)$. Note that (3.16) holds in the strong sense in $\mathbb{R}^n \times (0, \infty)$ because of the global estimates (3.14) and (3.17). Indeed, by (3.17) and (3.16) it follows that

$$U_\sigma(\cdot, y), \mathfrak{L}_\mathbf{A} U_\sigma(\cdot, y) \in L^p(\mathbb{R}^n) \quad \text{for all } y > 0,$$

hence (3.14) gives that $U_\sigma(\cdot, y) \in W^{2,p}(\mathbb{R}^n)$ for all $y > 0$. Then we also have that $\Delta U_\sigma \in L^p_{\text{loc}}(\mathbb{R}^n \times (0, \infty))$ and therefore $U_\sigma \in W^{2,p}_{\text{loc}}(\mathbb{R}^n \times (0, \infty))$. Then from (3.16), The Sobolev embeddings, the local estimates in Theorem 3.10, and a bootstrapping argument it follows that

$$(3.20) \quad U_\sigma \in W^{2,q}_{\text{loc}}(\mathbb{R}^n \times (0, \infty)) \quad \text{for all } 1 < q < \infty.$$

And Morrey's inequality implies that $U_\sigma \in C^{1,\alpha}(\mathbb{R}^n \times (0, \infty))$ for all $0 < \alpha < 1$.

Proposition 3.11. *Let $\mathfrak{L}_\mathbf{A}$ be given by (1.2) satisfy (1.1) with*

$$\sup_{1 \leq i, j \leq n} \|a^{ij}\|_* < \varepsilon_0$$

for ε_0 as in Lemma 3.9, so that for ω given by (1.3) $\mathfrak{L}_\mathbf{A}$ is of type ω . Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set, suppose $u \in \mathcal{D}(\mathfrak{L}_\mathbf{A}^\sigma) \subset L^p(\mathbb{R}^n)$, for some $0 < \sigma < 1$ and $1 < p < \infty$, and suppose that u satisfies $\mathfrak{L}_\mathbf{A}^\sigma u = 0$ in Ω . Then there exists a function V in $\Omega \times \mathbb{R}$ satisfying

$$V, V_z \in L^p_{\text{loc}}(\Omega \times (\mathbb{R})) \quad \text{and} \quad V \in W^{2,q}_{\text{loc}}(\Omega \times (\mathbb{R} \setminus \{0\}))$$

for all $1 < q < \infty$, such that $V(\cdot, 0) = u$ in $L^p(\Omega)$ and V is a strong solution of the problem

$$(3.21) \quad -\mathfrak{L}_\mathbf{A} V(x, z) + z^{2-\frac{1}{\sigma}} \frac{\partial^2}{\partial z^2} V(x, z) = 0$$

in $\Omega \times \mathbb{R} \setminus \{0\}$ such that $V(\cdot, z) \rightarrow u$ as $z \rightarrow 0$ in $L^p(\Omega)$.

Proof. Let U be given by (3.15), and set $\tilde{V}(x, z) = U\left(x, 2\sigma z^{\frac{1}{2\sigma}}\right) = U(x, y)$, performing the change of variables $z = \left(\frac{y}{2\sigma}\right)^{2\sigma}$ as in [3]. Equation (3.16) becomes (3.21), which holds in the strong sense in $\mathbb{R}^n \times (0, \infty)$ since, by (3.20), we have that $\tilde{V} \in W^{2,q}_{\text{loc}}(\mathbb{R}^n \times (0, \infty))$ for all $1 < q < \infty$.

By (3.18) we have that

$$\tilde{V}_z(\cdot, z) = (2\sigma)^{2\sigma-1} y^{1-2\sigma} U_y(\cdot, y) \rightarrow \frac{(2\sigma)^{2\sigma} \Gamma(-\sigma)}{4\sigma \Gamma(\sigma)} \mathfrak{L}_\mathbf{A}^\sigma u$$

in $L^p(\mathbb{R}^n)$ as $z \rightarrow 0^+$. Now we set

$$V(x, z) = \begin{cases} V(x, z) & x \in \Omega, z \geq 0 \\ V(x, -z) & x \in \Omega, z < 0 \end{cases},$$

and since from $\mathfrak{L}_{\mathbf{A}}^\sigma u \equiv 0$ in Ω , it follows that $V_z(\cdot, z) \rightarrow 0$ in $L^p(\Omega)$ as $z \rightarrow 0$. Hence $\frac{\partial V}{\partial z}$ extends to $\Omega \times \mathbb{R}$ as an L^p function on any bounded strip $\Omega \times (-N, N)$. By (3.17) and Theorem 2.1 we have that $V \in L_{\text{loc}}^p(\Omega \times (\mathbb{R}))$ and that $V(\cdot, z) \rightarrow u$ as $z \rightarrow 0$ in $L^p(\Omega)$. \square

Theorem 1.5 is now a consequence of this result.

Proof of Theorem 1.5. Let $U = V_\sigma \in W_{\text{loc}}^{2,q}(\Omega \times (\mathbb{R} \setminus \{0\}))$ be as in Proposition 3.11. By Theorem 2.1, (3.21), and the hypothesis $u \in \mathcal{D}(\mathfrak{L}_{\mathbf{A}})$, it follows that

$$\lim_{z \rightarrow 0} z^{2-\frac{1}{\sigma}} U_{zz}(\cdot, z) = \lim_{y \rightarrow 0} \mathfrak{L}_{\mathbf{A}} U(\cdot, y) = \mathfrak{L}_{\mathbf{A}} u \in L^p(\Omega)$$

where the limit is in $L^p(\mathbb{R}^n)$. From $0 < \sigma < \frac{p}{p+1}$ it follows that

$$\left(2 - \frac{1}{\sigma}\right) \frac{p}{p-1} < 1,$$

Let $1 < r < p$ such that $r \left(2 - \frac{1}{\sigma}\right) \frac{p}{p-r} < 1$. For each $N > 0$ if $\Omega' \Subset \Omega$ we have

$$\begin{aligned} & \left(\int_{\Omega'} \int_{-N}^N |U_{zz}(x, z)|^r dz dx \right)^{\frac{1}{r}} \\ & \leq \left(\int_{\Omega'} \int_{-N}^N \left| z^{2-\frac{1}{\sigma}} U_{zz}(x, z) \right|^p dz dx \right)^{\frac{1}{p}} \left(\int_{\Omega'} \int_{-N}^N z^{-(2-\frac{1}{\sigma})r \frac{p}{p-r}} dz dx \right)^{\frac{p-r}{rp}} \\ & \leq C_{N,\sigma,p,\Omega'} \left(\int_{\Omega'} \int_0^N \left| z^{2-\frac{1}{\sigma}} U_{zz}(x, z) \right|^p dz dx \right)^{\frac{1}{p}} < \infty. \end{aligned}$$

Thus, U_{zz} extends as an L_{loc}^r function in all bounded strips $\Omega \times (-N, N)$, $N > 0$. By proposition 3.11 we already have that $U_z \in L_{\text{loc}}^r(\Omega \times (-N, N))$. Moreover, by the local estimates in Theorem 3.10 we have that $U(\cdot, z) \in W_{\text{loc}}^{2,p}(\Omega) \subset W_{\text{loc}}^{2,r}(\Omega)$ for all z , with locally uniform bounds for bounded z . Hence $\Delta U \in L_{\text{loc}}^r(\Omega \times \mathbb{R})$ and consequently $U \in W_{\text{loc}}^{2,r}(\Omega \times \mathbb{R})$. Then by Theorem 3.10, the Sobolev's embedding, and a bootstrapping argument we conclude that

$$U \in W_{\text{loc}}^{2,q}(\Omega \times \mathbb{R}) \quad \text{for all } 1 < q < \infty.$$

Then Morrey's inequality implies that $U \in C^{1,\alpha}(\Omega \times \mathbb{R})$ for all $0 < \alpha < 1$. Since $u(z) = U(z, 0)$ we conclude that $u \in C^{1,\alpha}(\Omega)$. \square

4. APPENDIX

Let \mathcal{X} be a Banach space; $\mathfrak{L}(\mathcal{X})$ denotes the algebra of bounded linear operators on \mathcal{X} . Given a linear operator T on \mathcal{X} , the resolvent set $\rho(T)$ is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ is one to one and $R_T(\lambda) = (T - \lambda)^{-1} \in \mathfrak{L}(\mathcal{X})$; $R_T(\lambda)$ is called the resolvent the operator of T at λ . The spectrum of T , $\sigma(T)$ is the complement of $\rho(T)$ in \mathbb{C} , together with ∞ if T is not bounded.

We consider closed operators $T : \mathcal{D}(T) \subset \mathcal{X} \rightarrow \mathcal{X}$ where \mathcal{X} is a Banach space. Such operators have a holomorphic functional calculus. We denote by $\mathcal{CL}(\mathcal{X})$ the set of all closed operators on \mathcal{X} ; note that $\mathcal{L}(\mathcal{X}) \subset \mathcal{CL}(\mathcal{X})$.

4.1. Non-symmetric Dirichlet forms. Dirichlet forms can be defined in general Hilbert spaces, but for our applications it suffices to consider L^2 spaces. Specifically, let X be a locally compact metric space and μ is a σ -finite positive Radon measure on X such that $\text{support } \mu = X$. We will work on the real Hilbert space $L^2(X, \mu)$ with the usual L^2 -inner product $\langle \cdot, \cdot \rangle$, and in this context $\|f\|$ denotes the L^2 -norm $\langle f, f \rangle^{1/2}$. The basics of non-symmetric Dirichlet forms presented here can be found in chapter 1 of [21]; for symmetric Dirichlet forms see [12].

A bilinear form \mathcal{E} with domain $\mathcal{D}[\mathcal{E}] = \mathcal{F} \subset L^2(X, \mu)$ is a function $\mathcal{E} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$ which is linear in each variable separately.

Definition 4.1. A bilinear form \mathcal{E} on $\mathcal{F} \subset L^2(X, \mu)$ is a *(semi-)Dirichlet form* on $L^2(X, \mu)$ if \mathcal{F} is a dense subspace of $L^2(X, \mu)$ and the following conditions are satisfied:

- (I) \mathcal{E} is *lower bounded*: There exists a nonnegative constant α_0 such that

$$\mathcal{E}_{\alpha_0}(u, u) \geq 0 \quad \text{for all } u \in \mathcal{F},$$

where $\mathcal{E}_{\alpha_0}(u, v) = \mathcal{E}(u, v) + \alpha_0 \langle u, v \rangle$.

- (II) \mathcal{E} *satisfies the sector condition*: There exists a constant $K \geq 1$ such that

$$|\mathcal{E}(u, v)| \leq K \mathcal{E}_{\alpha_0}(u, u)^{1/2} \mathcal{E}_{\alpha_0}(v, v)^{1/2} \quad \text{for all } u, v \in \mathcal{F}.$$

- (III) \mathcal{F} is a Hilbert space relative to the inner product

$$\mathcal{E}_{\alpha}^{(s)}(u, v) = \frac{1}{2} (\mathcal{E}_{\alpha}(u, v) + \mathcal{E}_{\alpha}(v, u)) \quad \text{for all } \alpha > \alpha_0.$$

- (IV) \mathcal{E} *satisfies the Markov property*: for all $u \in \mathcal{F}$ and $a \geq 0$, then $u \wedge a \in \mathcal{F}$,

$$\mathcal{E}(u \wedge a, u - u \wedge a) \geq 0.$$

Note that for $\alpha > \alpha_0$ we have, with K as in (III) and $K_\alpha = K + \frac{\alpha}{\alpha - \alpha_0}$,

$$|\mathcal{E}_\alpha(u, v)| \leq K_\alpha \mathcal{E}_\alpha(u, u)^{1/2} \mathcal{E}_\alpha(v, v)^{1/2} \quad \text{for all } u, v \in \mathcal{F}.$$

In particular, \mathcal{E}_α and \mathcal{E}_β determine equivalent metrics for any fixed $\alpha, \beta > 0$.

When $\alpha_0 = 0$ in the above definition we say that \mathcal{E} is a *nonnegative Dirichlet form*. If a nonnegative Dirichlet form \mathcal{E} also satisfies

$$(4.1) \quad (u - u \wedge a, u \wedge a) \geq 0 \quad \text{for all } u \in \mathcal{F} \text{ and } a \geq 0,$$

then we say that \mathcal{E} is a *non-symmetric Dirichlet form*. If a non-symmetric Dirichlet form satisfies $\mathcal{E}(u, v) = \mathcal{E}(v, u)$ for all $u, v \in \mathcal{F}$ then \mathcal{E} is called *symmetric Dirichlet form*.

The framework of Dirichlet forms includes the first two applications that we will present in this work. Associated to each Dirichlet form \mathcal{E} there is an operator $-L_\mathcal{E}$ which is the generator of a strongly continuous semigroup $e^{-tL_\mathcal{E}}$. In fact this result is true for forms that are just closed, the following theorem can be found in [21] (Theorem 1.1.2).

Theorem 4.2. *Suppose \mathcal{E} is a bilinear form with dense domain $\mathcal{F} \subset L^2(X, \mu)$, and which satisfies (I), (II), and (III) from Definition 4.1. Then there exist strongly continuous semigroups $\{T_t\}_{t>0}$ and $\{\widehat{T}_t\}_{t>0}$ on $L^2(X, \mu)$ such that $\|T_t\| \leq e^{\alpha_0 t}$, $\|\widehat{T}_t\| \leq e^{\alpha_0 t}$, $\langle T_t f, g \rangle = \langle f, \widehat{T}_t g \rangle$ and whose resolvents*

$$G_\alpha = \int_0^\infty e^{-\alpha t} T_t dt \quad \text{and} \quad \widehat{G}_\alpha = \int_0^\infty e^{-\alpha t} \widehat{T}_t dt$$

satisfy

$$\mathcal{E}_\alpha(G_\alpha f, u) = \langle f, u \rangle = \mathcal{E}_\alpha(u, \widehat{G}_\alpha f),$$

for all $f \in L^2(X, \mu)$, $u \in \mathcal{F}$, and $\alpha > 0$. Moreover, $T_t = e^{-tL_\mathcal{E}}$ and $\widehat{T}_t = e^{-t\widehat{L}_\mathcal{E}}$ where the generators $L_\mathcal{E}$ and $\widehat{L}_\mathcal{E}$, also called the associated operator to \mathcal{E} and the associated adjoint operator to \mathcal{E} , respectively, have domains $\mathcal{D}(L_\mathcal{E}) \subset \mathcal{F}$ and $\mathcal{D}(\widehat{L}_\mathcal{E}) \subset \mathcal{F}$ which are dense in $L^2(X, \mu)$; for all $\alpha > \alpha_0$ and $f \in L^2(X, \mu)$ we have

$$G_\alpha f = (\alpha - \alpha_0 + L_\mathcal{E})^{-1} f \quad \text{and} \quad \widehat{G}_\alpha f = (\alpha - \alpha_0 + \widehat{L}_\mathcal{E})^{-1} f;$$

with the bounds

$$(4.2) \quad \|G_\alpha\| \leq \frac{1}{\alpha - \alpha_0} \quad \text{and} \quad \|\widehat{G}_\alpha\| \leq \frac{1}{\alpha - \alpha_0}.$$

Finally, for all $u \in \mathcal{D}(L_\mathcal{E})$, $v \in \mathcal{D}(\widehat{L}_\mathcal{E})$, and $f \in \mathcal{F}$ we have the identities

$$(4.3) \quad \langle L_\mathcal{E} u, f \rangle = \mathcal{E}_{\alpha_0}(u, f), \quad \text{and} \quad \langle f, \widehat{L}_\mathcal{E} v \rangle = \mathcal{E}_{\alpha_0}(f, v).$$

In this case we also have $\langle L_{\mathcal{E}}u, v \rangle = \mathcal{E}_{\alpha_0}(u, v) = \langle u, \widehat{L}_{\mathcal{E}}v \rangle$.

Note that since \mathcal{F} is dense in $L^2(X, \mu)$, the operators $L_{\mathcal{E}}$ and $\widehat{L}_{\mathcal{E}}$ are characterized by (4.3). That is, if $h \in L^2(X, \mu)$ and $\langle h, f \rangle = \mathcal{E}_{\alpha_0}(u, f)$ for some $u \in \mathcal{D}(L_{\mathcal{E}})$ and all $f \in \mathcal{F}$, then $L_{\mathcal{E}}u = h$. Moreover, because of (4.3) and the completeness assumption (III) we have that $L_{\mathcal{E}}$ is closed; i.e. if $u_n \in \mathcal{D}(L_{\mathcal{E}})$, $u_n \rightarrow u$ in $L^2(X, \mu)$, and $f_n = L_{\mathcal{E}}u_n \rightarrow f$ in $L^2(X, \mu)$, then $u \in \mathcal{D}(L_{\mathcal{E}})$ and $Lu = f$. Equivalently, $L_{\mathcal{E}}$ is closed if and only if $\mathcal{D}(L_{\mathcal{E}})$ is a Banach space with the norm $\|u\|_{\mathcal{D}(L_{\mathcal{E}})} = \|u\| + \mathcal{E}(u, u)^{\frac{1}{2}} \approx \mathcal{E}_1(u, u)^{\frac{1}{2}}$, this condition is guaranteed by (III). Similar statements apply to $\widehat{L}_{\mathcal{E}}$.

4.2. Sectorial operators and their calculus. All the operators we consider in our present applications are *sectorial* operators. This type of operators was first introduced by Kato [15], but here we adopt the more general definition in which we do not require the operator to be given by a sectorial form. Our definition is precisely that of operators of *type* ω as introduced by McIntosh [17], which was generalized as sectorial operators more recently to include Banach spaces (see [14, 1, 2] and references within).

Given $0 \leq \omega < \pi$ we denote by Σ_{ω} the open complex sector

$$\Sigma_{\omega} = \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \omega\}.$$

Definition 4.3. Given $0 \leq \omega < \pi$, an operator T on a Banach space \mathcal{X} is said to be of *type* ω , or *sectorial* of angle ω , if T closed and densely defined in \mathcal{X} , $\sigma(T) \subset \overline{\Sigma_{\omega}} \cup \{\infty\}$, and for each $\theta \in (\omega, \pi]$ there exists a constant $c_{\theta} > 0$ such that

$$\|R_T(z)\| = \|(z - T)^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq \frac{c_{\theta}}{|z|}$$

for all non-zero $z \notin \Sigma_{\theta}$.

If T is a sectorial of angle ω on \mathcal{X} with $0 \leq \omega < \pi/2$, the natural approach to establishing a holomorphic functional calculus and defining $\varphi(T)$ for $\varphi \in H^{\infty}(\Sigma_{\mu})$ is to first consider φ in the smaller class $H_0^{\infty}(\Sigma_{\mu})$, given by

$$H_0^{\infty}(\Sigma_{\mu}) = \left\{ \varphi \in H(\Sigma_{\mu}) : \exists c, s > 0 \quad |\varphi(z)| \leq \frac{c|z|^s}{(1 + |z|)^{-2s}}, \forall z \in \Sigma_{\mu} \right\}.$$

First, the semigroup e^{-zT} existence may be established by the Cauchy integral identity

$$e^{-zT} = \int_{\Gamma_{\alpha}} e^{-z\zeta} R_T(\zeta) d\zeta$$

where Γ_{α} is the boundary of Σ_{α} with positive orientation, and α is for any fixed angle such that $\omega < \alpha < \pi/2 - \arg z$. This semigroup is contractive ($\|e^{-zT}\| \leq 1$) and holomorphic in the sector $\Sigma_{\pi/2-\omega}$. Then we can write

an integral representation of $\varphi(T)$ for any $\varphi \in H_0^\infty(\Sigma_\mu)$, with $\omega < \theta < \nu < \min(\mu, \pi/2)$, namely:

$$(4.4) \quad \varphi(T) = \int_{\Gamma_{\pi/2-\theta}} e^{-zT} \eta(z) dz,$$

where

$$(4.5) \quad \eta(z) = \frac{1}{2\pi i} \int_{\gamma_\nu(z)} e^{\zeta z} \varphi(\zeta) d\zeta$$

with $\gamma_\nu(z) = \mathbb{R}^+ e^{i \operatorname{sign}(\operatorname{Im}(z))\nu}$. Note that

$$|\eta(z)| \lesssim \min(1, |z|^{-s-1}), \quad z \in \Gamma_{\pi/2-\theta},$$

so the representation (4.4) converges in \mathcal{X} , and we have the bound

$$(4.6) \quad \|\varphi(T)f\| \leq C \|\varphi\|_\infty \|f\|, \quad f \in H_0^\infty(\Sigma_\mu),$$

where $\|f\|$ denotes the norm of f in \mathcal{X} .

Now, if T is an operator of type ω as above, then T has an H^∞ functional calculus and (4.6) extends to all of $H^\infty(\Sigma_\mu)$ and also to holomorphic functions of polynomial growth (see also [17, 6, 14]). In particular, this approach allows us to define (fractional) powers T^σ of T for any $\sigma \in \mathbb{R}$. Of course, these operators will not in general be bounded if T is not bounded. The following is a resolution of fractional powers T^σ , for $\sigma > 0$:

$$(4.7) \quad T^\sigma x_0 = \frac{1}{\Gamma(-\sigma)} \int_0^\infty (e^{-tT} - 1) x_0 \frac{dt}{t^{1+\sigma}}.$$

This is the resolution we adopt in Theorem 2.1. By the Spectral Mapping Theorem it follows that if T is sectorial of angle ω , then T^σ is sectorial of angle $\sigma\omega$ for any $0 < \sigma \leq 1$, $\overline{\mathcal{D}(T)} = \mathcal{D}(T^\sigma) \supset \mathcal{D}(T)$, and $\mathcal{N}(T^\sigma) = \mathcal{N}(T)$, where $\mathcal{N}(T)$ denotes the kernel of T (see Proposition 3.1.1. in [14]).

A real bilinear form \mathcal{E} with domain $\mathcal{D}[\mathcal{E}] \subset (L^2(X, \mu), \mathbb{R})$ has a natural extension as a complex sesquilinear form $\tilde{\mathcal{E}}$ with domain $\mathcal{D}[\tilde{\mathcal{E}}] = \mathcal{D}[\mathcal{E}] + i\mathcal{D}[\mathcal{E}] \subset (L^2(X, \mu), \mathbb{C})$ by setting

$$(4.8) \quad \begin{aligned} \tilde{\mathcal{E}}(f_1, f_2) &= \tilde{\mathcal{E}}(g_1 + ih_1, g_2 + ih_2) \\ &= \mathcal{E}(g_1, g_2) + \mathcal{E}(h_1, h_2) + i\mathcal{E}(h_1, g_2) - i\mathcal{E}(g_1, h_2) \end{aligned}$$

where $g_i = \operatorname{Re} f_i$ and $h_i = \operatorname{Im} f_i$, $i = 1, 2$. Note that \mathcal{E} is indeed the restriction of $\tilde{\mathcal{E}}$ to $\mathcal{D}[\mathcal{E}] \subset (L^2(X, \mu), \mathbb{R})$. Thus, if \mathcal{E} is as in the previous corollary, the sesquilinear form $\tilde{\mathcal{E}}$ is *accretive*, that is, $\operatorname{Re} \tilde{\mathcal{E}}(f, f) \geq 0$. Moreover, if K is the constant from condition (II) in Definition 4.1, $\tilde{\mathcal{E}}$ is sectorial with the same constant:

$$\left| \operatorname{Im} \tilde{\mathcal{E}}(f, f) \right| \leq K \operatorname{Re} \tilde{\mathcal{E}}(f, f).$$

The operator $L_{\tilde{\mathcal{E}}}$ associated to this sesquilinear form (see 1.2.3 in [22]), and its corresponding adjoint operator $\widehat{L}_{\tilde{\mathcal{E}}}$ are the generators of a holomorphic semigroup in the sector $\Sigma_{\arctan(1/K)}$, see Theorem 1.53 in [22] for a proof of the next result.

Theorem 4.4. *Let \mathcal{E} is a nonnegative bilinear form with dense domain in $L^2(X, \mu)$, satisfying (I), (II), and (III), and let $\tilde{\mathcal{E}}$ be the sesquilinear extension (4.8). Then the associated operators $-L_{\tilde{\mathcal{E}}}$ and $-\widehat{L}_{\tilde{\mathcal{E}}}$ generate strongly continuous semigroups $e^{-tL_{\tilde{\mathcal{E}}}}$ and $e^{-t\widehat{L}_{\tilde{\mathcal{E}}}}$, $t \geq 0$, on $L^2(\mathcal{X}, \mu)$. These semigroups are holomorphic on the sector $\Sigma_{\arctan(1/K)}$ and the operators $e^{-zL_{\tilde{\mathcal{E}}}}$, $e^{-z\widehat{L}_{\tilde{\mathcal{E}}}}$ are contraction operators, i.e. $\|e^{-zL_{\tilde{\mathcal{E}}}}\| \leq 1$ and $\|e^{-z\widehat{L}_{\tilde{\mathcal{E}}}}\| \leq 1$, for all $z \in \Sigma_{\arctan(1/K)}$.*

Proposition 4.5. *Let \mathcal{E} is a nonnegative bilinear form with dense domain \mathcal{F} in $L^2(\mathcal{X}, \mu)$, satisfying (I), (II), and (III), and let $\tilde{\mathcal{E}}$ be the sesquilinear extension (4.8). Then the associated operator $L_{\mathcal{E}}$ ($\widehat{L}_{\mathcal{E}}$) is the restriction of the operator $L_{\tilde{\mathcal{E}}}$ ($\widehat{L}_{\tilde{\mathcal{E}}}$) in the sense that $\mathcal{D}(L_{\mathcal{E}}) = \text{Re } \mathcal{D}(L_{\tilde{\mathcal{E}}})$ ($\mathcal{D}(\widehat{L}_{\mathcal{E}}) = \text{Re } \mathcal{D}(\widehat{L}_{\tilde{\mathcal{E}}})$), and $L_{\mathcal{E}}(\text{Re } u) = \text{Re}(L_{\tilde{\mathcal{E}}}u)$ ($\widehat{L}_{\mathcal{E}}(\text{Re } u) = \text{Re}(\widehat{L}_{\tilde{\mathcal{E}}}u)$) for all $u \in \mathcal{D}(L_{\mathcal{E}})$ ($\mathcal{D}(\widehat{L}_{\mathcal{E}})$).*

Proof. Suppose $u \in \mathcal{D}(L_{\tilde{\mathcal{E}}})$, then for all $v \in \mathcal{D}[\tilde{\mathcal{E}}] = \mathcal{D}[\mathcal{E}] + i\mathcal{D}[\mathcal{E}]$

$$\langle L_{\tilde{\mathcal{E}}}u, v \rangle = \int_{\mathcal{X}} (L_{\tilde{\mathcal{E}}}u) \bar{v} \, d\mu = \tilde{\mathcal{E}}(u, v).$$

Since $\mathcal{D}[\tilde{\mathcal{E}}] \supset \mathcal{D}[\mathcal{E}] = \text{Re } \mathcal{D}[\tilde{\mathcal{E}}]$, we have that for $v \in \mathcal{D}[\mathcal{E}]$

$$\text{Re} \langle L_{\tilde{\mathcal{E}}}u, v \rangle = \int_{\mathcal{X}} \text{Re}(L_{\tilde{\mathcal{E}}}u) v \, d\mu = \text{Re } \tilde{\mathcal{E}}(u, v) = \mathcal{E}(\text{Re } u, v).$$

Thus $\text{Re } u \in \mathcal{D}(L_{\mathcal{E}})$ and $L_{\mathcal{E}}(\text{Re } u) = \text{Re}(L_{\tilde{\mathcal{E}}}u)$. Similarly, if $u \in \mathcal{D}(L_{\mathcal{E}})$, and $v \in \mathcal{D}[\tilde{\mathcal{E}}]$, write $v = f + ig$ where $f = \text{Re } v$ and $g = \text{Im } v$, then

$$\begin{aligned} \langle L_{\mathcal{E}}u, v \rangle &= \langle L_{\mathcal{E}}u, f \rangle - i \langle L_{\mathcal{E}}u, g \rangle \\ &= \int_{\mathcal{X}} (L_{\mathcal{E}}u) f \, d\mu - i \int_{\mathcal{X}} (L_{\mathcal{E}}u) g \, d\mu \\ &= \mathcal{E}(u, f) - i\mathcal{E}(u, g) = \tilde{\mathcal{E}}(u, v). \end{aligned}$$

So $u \in \mathcal{D}(L_{\tilde{\mathcal{E}}})$ and $L_{\mathcal{E}}u = L_{\tilde{\mathcal{E}}}u$. The proof for the operators $\widehat{L}_{\mathcal{E}}$ and $\widehat{L}_{\tilde{\mathcal{E}}}$ is similar. \square

As a consequence of the previous proposition and Theorem 4.4 we have that if \mathcal{E} is a nonnegative bilinear form as in the proposition then the operators $L_{\mathcal{E}}$ and $L_{\tilde{\mathcal{E}}}$ generate strongly continuous semigroups in $(-\infty, 0]$ and holomorphic contractive semigroups $e^{-zL_{\mathcal{E}}}$, $e^{-z\hat{L}_{\mathcal{E}}}$ on the sector $\Sigma_{\arctan(1/K)}$. In turn, standard results imply that the operators $L_{\mathcal{E}}$ and $\hat{L}_{\mathcal{E}}$ are sectorial of angle $\frac{\pi}{2} - \arctan(1/K)$, see for example Theorem II.4.6 in [9] for a proof. We collect these facts in the following corollary.

Corollary 4.6. *Let \mathcal{E} is a nonnegative bilinear form with dense domain in $L^2(X, \mu)$, satisfying (I), (II), and (III) from Definition 4.1. Then the associated operators $L_{\tilde{\mathcal{E}}}$ and $\hat{L}_{\tilde{\mathcal{E}}}$ are sectorial of angle $\frac{\pi}{2} - \arctan(1/K)$, where K is then constant in (II). Moreover, $-L_{\tilde{\mathcal{E}}}$ and $-\hat{L}_{\tilde{\mathcal{E}}}$ generate strongly continuous semigroups $e^{-tL_{\tilde{\mathcal{E}}}}$ and $e^{-t\hat{L}_{\tilde{\mathcal{E}}}}$, $t \geq 0$, on $L^2(X, \mu)$. These semigroups are holomorphic on the sector $\Sigma_{\arctan(1/K)}$ and the operators $e^{-zL_{\tilde{\mathcal{E}}}}$, $e^{-z\hat{L}_{\tilde{\mathcal{E}}}}$ are contraction operators.*

REFERENCES

1. Lashi Bandara and Alan McIntosh, *Operator theory - spectra and functional calculi*, Lecture Notes (2010), 1–74.
2. Charles Batty, *Unbounded operators: functional calculus, generation, perturbations*, Extracta Math. **24** (2009), no. 2, 99–133. MR 2603795
3. Luis Caffarelli and Luis Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7-9, 1245–1260. MR 2354493 (2009k:35096)
4. Filippo Chiarenza, Michele Frasca, and Placido Longo, *Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients*, Ricerche Mat. **40** (1991), no. 1, 149–168. MR 1191890
5. ———, *$W^{2,p}$ -solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. **336** (1993), no. 2, 841–853. MR 1088476
6. M. Cowling, I. Doust, A. McIntosh, and A. Yagi, *Banach space operators with a bounded H^∞ functional calculus*, J. Austral. Math. Soc. Ser. A **60** (1996), no. 1, 51–89. MR 1364554 (97d:47023)
7. Hongjie Dong and Doyoon Kim, *On the impossibility of W_p^2 estimates for elliptic equations with piecewise constant coefficients*, J. Funct. Anal. **267** (2014), no. 10, 3963–3974. MR 3266252
8. Xuan Thinh Duong and Li Xin Yan, *Bounded holomorphic functional calculus for non-divergence form differential operators*, Differential Integral Equations **15** (2002), no. 6, 709–730. MR 1893843
9. Klaus-Jochen Engel and Rainer Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000, With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. MR 1721989
10. E. B. Fabes, C. Kenig, and R. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116. MR MR643158 (84i:35070)

11. C. Fefferman and D. H. Phong, *Subelliptic eigenvalue problems*, Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983, pp. 590–606. MR 730094
12. Masatoshi Fukushima, Yoichi Oshima, and Masayoshi Takeda, *Dirichlet forms and symmetric Markov processes*, extended ed., de Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter & Co., Berlin, 2011. MR 2778606
13. David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR 1814364
14. M. Haase, *The functional calculus for sectorial operators*, Operator Theory: Advances and Applications, vol. 169, Birkhäuser Verlag, Basel, 2006. MR 2244037 (2007j:47030)
15. Tosio Kato, *Note on fractional powers of linear operators*, Proc. Japan Acad. **36** (1960), 94–96. MR 0121666
16. C. Kenig, H. Koch, J. Pipher, and T. Toro, *A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations*, Adv. Math. **153** (2000), no. 2, 231–298. MR 1770930 (2002f:35071)
17. A. McIntosh, *Operators which have an H_∞ functional calculus*, Miniconference on operator theory and partial differential equations (North Ryde, 1986), Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 14, Austral. Nat. Univ., Canberra, 1986, pp. 210–231. MR MR912940 (88k:47019)
18. Nicholas Miller, *Weighted Sobolev spaces and pseudodifferential operators with smooth symbols*, Trans. Amer. Math. Soc. **269** (1982), no. 1, 91–109. MR 637030
19. D. D. Monticelli, S. Rodney, and R. L. Wheeden, *Harnack’s inequality and Hölder continuity for weak solutions of degenerate quasilinear equations with rough coefficients*, Nonlinear Anal. **126** (2015), 69–114. MR 3388872
20. Charles B. Morrey, Jr., *Multiple integrals in the calculus of variations*, Classics in Mathematics, Springer-Verlag, Berlin, 2008, Reprint of the 1966 edition [MR0202511]. MR 2492985
21. Yoichi Oshima, *Semi-Dirichlet forms and Markov processes*, De Gruyter Studies in Mathematics, vol. 48, Walter de Gruyter & Co., Berlin, 2013. MR 3060116
22. E. M. Ouhabaz, *Analysis of heat equations on domains*, London Mathematical Society Monographs Series, vol. 31, Princeton University Press, Princeton, NJ, 2005. MR MR2124040 (2005m:35001)
23. Cristian Rios, *The L^p Dirichlet problem and nondivergence harmonic measure*, Trans. Amer. Math. Soc. **355** (2003), no. 2, 665–687. MR 1932720
24. Scott Rodney, *A degenerate Sobolev inequality for a large open set in a homogeneous space*, Trans. Amer. Math. Soc. **362** (2010), no. 2, 673–685. MR 2551502
25. Eric T. Sawyer and Richard L. Wheeden, *Hölder continuity of weak solutions to subelliptic equations with rough coefficients*, Mem. Amer. Math. Soc. **180** (2006), no. 847, x+157. MR 2204824 (2007f:35037)
26. ———, *Degenerate Sobolev spaces and regularity of subelliptic equations*, Trans. Amer. Math. Soc. **362** (2010), no. 4, 1869–1906. MR 2574880
27. Pablo Raúl Stinga and José Luis Torrea, *Extension problem and Harnack’s inequality for some fractional operators*, Comm. Partial Differential Equations **35** (2010), no. 11, 2092–2122. MR 2754080 (2012c:35456)

INSTITUTO DE MATEMÁTICA APLICADA DEL LITORAL, CCT, CONICET, SANTA FE,
ARGENTINA

E-mail address: `haimar@santafe-conicet.gov.ar`

INSTITUTO DE MATEMÁTICA APLICADA DEL LITORAL, CCT, CONICET, SANTA FE,
ARGENTINA

E-mail address: `gbeltritti@santafe-conicet.gov.ar`

INSTITUTO DE MATEMÁTICA APLICADA DEL LITORAL, CCT, CONICET, SANTA FE,
ARGENTINA

E-mail address: `vanagomez@santafe-conicet.gov.ar`

UNIVERSITY OF CALGARY, CALGARY, AB T2N1N4, CANADA

E-mail address: `crios@math.ucalgary.ca`